We study convergences of Mann and Ishikawa iteration processes for mappings of asymptotically quasi-nonexpansive type in Banach spaces.

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1. Introduction and preliminaries. Let $D$ be a nonempty subset of a real Banach space $X$ and $T: D \to D$ a nonlinear mapping. The mapping $T$ is said to be asymptotically quasi-nonexpansive (see [5]) if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} k_n = 0$ such that

$$\|T^n x - p\| \leq (1 + k_n) \|x - p\|$$

(1.1)

for all $x \in D$, $p \in F(T)$, and $n \in \mathbb{N}$. The mapping $T$ is said to be asymptotically nonexpansive (see [3]) if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} k_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + k_n) \|x - y\|$$

(1.2)

for all $x, y \in D$ and $n \in \mathbb{N}$. The mapping $T$ is said to be a mapping of asymptotically nonexpansive type [4] if

$$\limsup_{n \to \infty} \sup_{x \in D} (\|T^n x - T^n y\| - \|x - y\|) \leq 0$$

(1.3)

for any $y \in D$.


Recently, Liu [5] extended results of [2, 7] and gave the necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points of asymptotically quasi-nonexpansive mappings.
First, we introduce the concept of class of mappings of asymptotically quasinonexpansive type: the mapping \( T \) is said to be a mapping of asymptotically quasinonexpansive type if \( F(T) \neq \emptyset \) and

\[
\limsup_{n \to \infty} \sup_{x \in D} (\|T^n x - p\| - \|x - p\|) \leq 0 \quad \text{for any } p \in F(T).
\] (1.4)

**Remark 1.1.** If \( T \) is a mapping of asymptotically nonexpansive type with \( F(T) \neq \emptyset \), then \( T \) is a mapping of asymptotically quasinonexpansive type.

**Remark 1.2.** If \( D \) is bounded and \( T \) is an asymptotically quasinonexpansive mapping, then \( T \) is a mapping of asymptotically quasi-nonexpansive type.

In fact, if \( T \) is an asymptotically quasinonexpansive mapping, then there exists a sequence \( \{k_n\} \) in \([0, \infty)\) with \( \lim_{n \to \infty} k_n = 0 \) such that

\[
\|T^n x - p\| \leq (1 + k_n) \|x - p\|
\] (1.5)

for all \( x \in D, \ p \in F(T) \), and \( n \in \mathbb{N} \), which implies

\[
\sup_{x \in D} \{\|T^n x - T^n y\| - \|x - y\|\} \leq k_n \cdot \text{diam} \ D
\] (1.6)

for any \( y \in F(T) \) and \( n \in \mathbb{N} \). Hence

\[
\limsup_{n \to \infty} \sup_{x \in D} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \quad \text{for any } y \in F(T).
\] (1.7)

We observe from Remarks 1.1 and 1.2 that the class of mappings of asymptotically nonexpansive type is an intermediate class between the class of mappings of asymptotically quasi-nonexpansive type and that of mappings of asymptotically nonexpansive type with nonempty fixed-point sets. Let

\[
\begin{align*}
C_1 &= \{T : T : D \to D \text{ is a nonexpansive mapping}\}, \\
C_2 &= \{T : T : D \to D \text{ is a quasi-nonexpansive mapping}\}, \\
C_3 &= \{T : T : D \to D \text{ is an asymptotically nonexpansive mapping}\}, \\
C_4 &= \{T : T : D \to D \text{ is an asymptotically quasi-nonexpansive mapping}\}, \\
C_5 &= \{T : T : D \to D \text{ is a mapping of asymptotically nonexpansive type}\}, \\
C_6 &= \{T : T : D \to D \text{ is a mapping of asymptotically quasi-nonexpansive type}\}.
\end{align*}
\] (1.8)

Then we have the following implications:

\[
\begin{array}{cccc}
C_1 & \longrightarrow & C_2 & \longleftrightarrow \quad C_4 \\
\downarrow & & \downarrow & \downarrow \\
C_3 & \longrightarrow & C_4 & \longleftrightarrow \quad C_6.
\end{array}
\] (1.9)
In this paper, we are mainly interested in the problem of approximation of fixed points of the more general class of mappings of asymptotically quasi-nonexpansive type than that of asymptotically quasi-nonexpansive mappings. The purpose of this paper is to continue discussion concerning convergence of Mann and Ishikawa iteration processes for mappings of asymptotically quasi-nonexpansive type in Banach spaces. We give necessary and sufficient conditions for the Mann and Ishikawa iteration processes to converge to fixed points of mappings of asymptotically quasi-nonexpansive type. Further, we obtain extensions of various results obtained quite recently by Deng [1], Ghosh and Denath [2], Liu [5], and Tan and Xu [9, 10] to more general types of space as well as families of operators.

We say that a Banach space $X$ satisfies Opial’s condition [6] if, for each sequence $\{x_n\}$ in $X$ weakly convergent to a point $x$ and for all $y \neq x$,

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|.$$  

(1.10)

The examples of Banach spaces which satisfy Opial’s condition are Hilbert spaces, and all $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial’s condition [6].

Let $D$ be a nonempty closed convex subset of a Banach space $X$. Then $I - T$ is demiclosed at zero if, for any sequence $\{x_n\}$ in $D$, condition $x_n \to x$ weakly and $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$ implies $(I - T)x = 0$.

2. Main results. In this section, we establish some weak and strong convergences for mappings of asymptotically quasi-nonexpansive type in Banach spaces.

**Lemma 2.1.** Let $D$ be a nonempty subset of a normed space $X$ and let $T : D \to E$ be a mapping of asymptotically quasi-nonexpansive type. For two given real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$, let a sequence $\{x_n\}$ in $D$ be defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n,$$

$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad n = 1, 2, \ldots.$$  

(2.1)

If $p$ is a fixed point of $T$, then

(a) $\|x_{n+1} - p\| \leq \|x_n - p\| + (1 + \beta_n)\sup_{x \in D}(\|T^n x - p\| - \|x - p\|), \quad n = 1, 2, \ldots,$

(b) $\lim_{n \to \infty} \|x_n - p\|$ exists.

**Proof.** Let $p$ be a fixed point of $T$.

(a) From (2.1), we have

$$\|x_{n+1} - p\| \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|T^n y_n - p\|$$

$$\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n(\|T^n y_n - p\| - \|y - p\|) + \alpha_n\|y_n - p\|.$$
\[
\begin{align*}
\leq (1 - \alpha_n) \|x_n - p\| + (\|T^n y_n - p\| - \|y_n - p\|) \\
+ \alpha_n ((1 - \beta_n) \|x_n - p\| + \beta_n \|T^n x_n - p\|) \\
\leq \|x_n - p\| + (\|T^n y_n - p\| - \|y_n - p\|) \\
+ \beta_n (\|T^n x_n - p\| - \|x_n - p\|) \\
\leq \|x_n - p\| + (1 + \beta_n) \sup_{x \in D} (\|T^n x - p\| - \|x - p\|).
\end{align*}
\]  
(2.2)

(b) For \(m, n \in \mathbb{N}\), we have
\[
\begin{align*}
\|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + 2 \sup_{x \in D} (\|T^{n+m-1} x - p\| - \|x - p\|) \\
&\leq \|x_{n+m-1} - p\| + 2 \sup_{x \in D} (\|T^m x - p\| - \|x - p\|) \\
&\leq \|x_{n+m-2} - p\| + 4 \sup_{x \in D} (\|T^m x - p\| - \|x - p\|) \\
&\leq \cdots \leq \|x_n - p\| + 2n \sup_{x \in D} (\|T^m x - p\| - \|x - p\|).
\end{align*}
\]  
(2.3)

Hence, for \(n \in \mathbb{N}\),
\[
\limsup_{m \to \infty} \|x_m - p\| \leq \|x_n - p\| + 2n \limsup_{m \to \infty} (\|T^m x - p\| - \|x - p\|) \\
\leq \|x_n - p\|.
\]  
(2.4)

It follows that
\[
\limsup_{m \to \infty} \|x_m - p\| \leq \liminf_{n \to \infty} \|x_n - p\|.
\]  
(2.5)

Thus \(\lim_{n \to \infty} \|x_n - p\|\) exists.

**Lemma 2.2.** Let \(D\) and \(T\) be as in Lemma 2.1. For a given real sequence \(\{\alpha_n\}\) in \([0, 1]\), let a sequence \(\{x_n\}\) in \(D\) be defined by
\[
x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, \quad n = 1, 2, \ldots.
\]  
(2.6)

If \(p\) is a fixed point of \(T\), then
\[
\begin{align*}
(\text{a}) \quad \|x_{n+1} - p\| &\leq \|x_n - p\| + \sup_{x \in D} (\|T^n x - p\| - \|x - p\|), \quad n = 1, 2, \ldots, \\
(\text{b}) \quad \lim_{n \to \infty} \|x_n - p\| &\text{ exists.}
\end{align*}
\]

**Theorem 2.3.** Let \(X\) be a Banach space which satisfies Opial’s condition and let \(D\) be a weakly compact subset of \(X\). Let \(T\) and \(\{x_n\}\) be as in Lemma 2.1. Suppose that \(T\) has a fixed point, \(I - T\) is demiclosed at zero, and \(\{x_n\}\) is an approximating fixed-point sequence for \(T\), that is, \(\lim_{n \to \infty} \|x_n - T x_n\| = 0\). Then \(\{x_n\}\) converges weakly to a fixed point of \(T\).

**Proof.** First, we show that \(\omega_{w}(x_n) \subset F(T)\). Let \(x_{n_k} \to x\) weakly. By assumption, we have \(\lim_{n \to \infty} \|x_n - T x_n\| = 0\). Since \(I - T\) is demiclosed at zero,
Let $x \in F(T)$. By Opial’s condition, $\{x_n\}$ possesses only one weak limit point, that is, $\{x_n\}$ converges weakly to a fixed point of $T$. \hfill $\Box$

**Theorem 2.4.** Let $X$ be a Banach space which satisfies Opial’s condition and let $D$ be a weakly compact subset of $X$. Let $T$ and $\{x_n\}$ be as in Lemma 2.2. Suppose that $T$ has a fixed point, $I - T$ is demiclosed at zero, and $\{x_n\}$ is an approximating fixed-point sequence for $T$, that is, $\lim_{n \to \infty} \|x_n - T x_n\| = 0$. Then $\{x_n\}$ converges weakly to a fixed point of $T$.

**Remark 2.5.** Theorem 2.3 improves Theorem 2 of Deng [1] for mappings of asymptotically quasi-nonexpansive type. Theorem 2.4 generalizes Theorem 2.1 of Schu [8].

**Theorem 2.6.** Let $D$ be a closed subset of Banach space, let $T : D \to D$ be a mapping of asymptotically quasi-nonexpansive type, and $F(T)$ be nonempty closed set. For two given real sequences $\{\alpha_n\}$ and $\{\beta\}$ in $[0, 1]$, let the Ishikawa iterative sequence $\{x_n\}$ in $D$ be defined by (2.1). Then $\{x_n\}$ converges strongly to a fixed point of $T$ if and only if $\liminf_n d(x_n, F(T)) = 0$.

**Proof.** Let $\{x_n\}$ converge strongly to a point $z \in F(T)$. Then $\lim_n d(x_n, F(T)) = 0$. Conversely, suppose $\liminf_n d(x_n, F(T)) = 0$. From Lemma 2.1(a),

$$\|x_{n+1} - p\| \leq \|x_n - p\| + 2 \sup_{x \in D} (\|T^n x - p\| - \|x - p\|)$$

for any $n \in \mathbb{N}$ and $p \in F(T)$. Since $T$ is a mapping of asymptotically quasi-nonexpansive type, we have

$$\limsup_{n \to \infty} \left\{ \sup_{x \in D} (\|T^k x - p\| - \|x - p\|) \right\} \leq 0.$$  \hfill (2.7)

Hence, there exists a positive integer $n_0$ and a sequence $\{a_n\}$ of positive real numbers with $\lim_n a_n = 0$ such that

$$\sup_{k \geq n} \left\{ \sup_{x \in D} (\|T^k x - p\| - \|x - p\|) \right\} \leq a_n$$  \hfill (2.8)

for any $n \geq n_0$. Without loss of generality, we can assume that $a_n = 1/2n^2$. Hence,

$$\sup_{k \geq n} \left\{ \sup_{x \in D} (\|T^k x - p\| - \|x - p\|) \right\} \leq \frac{1}{2n^2}$$  \hfill (2.9)

for any $n \geq n_0$. It follows from (2.7) that

$$\|x_{n+1} - p\| \leq \|x_n - p\| + \frac{1}{n^2}$$  \hfill (2.10)
for all $n \geq n_0$, that is,
\begin{equation}
    d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + \frac{1}{n^2} \tag{2.12}
\end{equation}
for all $n \geq n_0$. Hence for $n, m \geq n_0$, we have
\begin{equation}
    d(x_{n+m}, F(T)) \leq d(x_n, F(T)) + \sum_{i=n}^{n+m-1} \frac{1}{i^2}. \tag{2.13}
\end{equation}

Using [10, Lemma 1, page 303], we obtain that $\lim_n d(x_n, F(T))$ exists, and it follows from $\liminf_n d(x_n, F(T)) = 0$ that $\lim_n d(x_n, F(T)) = 0$. Thus, $\lim_n d(x_n, F(T)) = 0$. For each $\varepsilon > 0$, there exists a natural number $m_0$ such that
\begin{equation}
    d(x_n, F(T)) < \frac{\varepsilon}{3} \tag{2.14}
\end{equation}
for all $n \geq m_0$. Then there exists a $p' \in F(T)$ such that $d(x_n, p') < \varepsilon/2$ for all $n \geq m_0$. If $n, m \geq m_0$, then
\begin{equation}
    d(x_n, x_m) \leq d(x_n, p') + d(p', x_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{2.15}
\end{equation}
This shows that $\{x_n\}$ is a Cauchy sequence in $D$. Let $\lim_n x_n = v \in D$. Since $F(T) \subset D$ is closed and $\lim_n d(x_n, F(T)) = 0$, we conclude that $v \in F(T)$. This completes the proof. \hfill \Box

As a consequence of Theorem 2.6, we obtain the following result.

**Theorem 2.7.** Let $D$ be a closed subset of Banach space, let $T : D \to D$ be a mapping of asymptotically quasi-nonexpansive type, and let $F(T)$ be a nonempty closed set. For a given sequence $\{\alpha_n\}$ in $[0, 1]$, let the Mann iterative sequence $\{x_n\}$ in $D$ be defined by (2.6). Then $\{x_n\}$ converges strongly to a fixed point of $T$ if and only if $\liminf_n d(x_n, F(T)) = 0$.

**Remark 2.8.** Theorems 2.6 and 2.7 extend corresponding results of Ghosh and Debnath [2], Liu [5], and Petryshyn and Williamson [7] from quasi-nonexpansive or asymptotically quasi-nonexpansive mapping to large class of non-Lipschitzian mappings.

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