INVARIANT SUBSPACES FOR POLYNOMIALLY COMPACT ALMOST SUPERDIAGONAL OPERATORS ON $l(p_i)$

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It is shown that almost superdiagonal, polynomially compact operators on the sequence space $l(p_i)$ have nontrivial, closed invariant subspaces if the nonlocally convex linear topology $\tau(p_i)$ is locally bounded.

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1. Introduction. The purpose of this paper is to show that almost superdiagonal, polynomially compact operators on the sequence space $l(p_i)$ have nontrivial, closed invariant subspaces if the nonlocally convex linear topology $\tau(p_i)$ is locally bounded. The proofs and arguments of this paper are stated within the framework of nonstandard analysis (see [4, Theorem 6.3 and Proposition 5.5]).

1.1. Preliminaries. Let $\{p_i\}_{i=1}^{\infty}$ be a sequence of real numbers such that $0 < p_i \leq 1$ for each $i \in \mathbb{N}_+$, the set of positive integers. Let

$$l(p_i) = \left\{ (\xi_i) \mid \sum_{i=1}^{\infty} |\xi_i|^{p_i} < \infty \right\},$$

(1.1)

where $\xi_i \in \mathbb{C}$, the complex numbers. Since $|\lambda + \beta|^p \leq |\lambda|^p + |\beta|^p$ and $|\lambda \beta|^p \leq \max(1,|\lambda|)|\beta|^p$ are valid for all $\lambda, \beta \in \mathbb{C}$ and $0 < p \leq 1$, it follows that $l(p_i)$ is a vector space over $\mathbb{C}$. Also,

$$\rho(p_i)(x,y) = \sum_{i=1}^{\infty} |\xi_i - \zeta_i|^{p_i},$$

(1.2)

where $x = (\xi_i)$ and $y = (\zeta_i)$, defines a translation invariant metric on $l(p_i)$. Let $\tau(p_i)$ denote the topology generated on $l(p_i)$ by $\rho(p_i)$. If $p_i = p \in (0,1]$ for all $i \in \mathbb{N}_+$, then we denote $l(p_i)$ by $l^p$ and $\tau(p_i)$ by $\tau_p$.

For $(l(p_i), \tau(p_i))$, the following facts are known:

(1) $(l(p_i), \tau(p_i))$ is a complete topological vector space;
(2) $(l(p_i), \tau(p_i))$ is a locally convex space if and only if $l(p_i) = l^1$;
(3) the following three conditions on $\{p_i\}_{i=1}^{\infty} \subset (0,1]$ are equivalent:
(a) \( \liminf p_i > 0 \),
(b) a subset \( B \) of \( l(p_i) \) is bounded in \( \tau(p_i) \) if and only if it is bounded in the metric \( \rho(p_i) \),
(c) \( (l(p_i), \tau(p_i)) \) is locally bounded, that is, there exists a \( \tau(p_i) \)-bounded neighborhood of 0.

(See [5, Lemmas 1 and 2, Theorems 5 and 6].)

Unless stated otherwise, it will be assumed that \( 0 < p_i \leq 1 \), for \( i \in \mathbb{N}_+ \), and \( 0 < p \leq \liminf p_i \).

The sequence \( \{e_i\} \) (where \( e_i = (\varepsilon_{ij}) \), \( \varepsilon_{ii} = 1 \), and \( \varepsilon_{ij} = 0 \) for \( i \neq j \)) will denote the natural Schauder basis for \( (l(p_i), \tau(p_i)) \) and \( \{\pi_i \mid i \in \mathbb{N}_+\}, \{P_j \mid j \in \mathbb{N}_+\} \), and \( \{E_j \mid j \in \mathbb{N}_+\} \) will denote the sequences of coordinate functionals, projections, and coordinate spaces, respectively, generated in \( l(p_i) \) by \( \{e_i\} \). Also, a \( \tau \) or \( \rho \), when used, will symbolize \( \tau(p_i) \) and \( \rho(p_i) \), respectively.

Let \( \mathcal{F}[l(p_i)] \) symbolize the collection of all functions mapping \( l(p_i) \) into \( l(p_i) \) and let \( \mathcal{L}(l(p_i)) \) designate the vector spaces of \( \tau(p_i) \)-continuous linear transformation and linear transformations on \( l(p_i) \), respectively. If \( T, U \in \mathcal{L}(l(p_i)) \), then \( TU \) denotes the composite map of \( T \) and \( U \). For \( n \in \mathbb{N} \), the set of natural numbers, and \( T \in \mathcal{L}(l(p_i)) \), define \( T^n \) in the usual manner, that is, \( T^0 = I \), the identity map, \( T^1 = T \), and \( T^n = TT^{n-1} \) for \( 1 \leq n \). If \( q(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^k \) is a polynomial over \( \mathbb{C} \), then we define \( q(T) = \sum_{k=0}^{\infty} c_k T^k \) for \( T \in \mathcal{L}(l(p_i)) \).

Let \( x \in l(p_i) \), which implies \( x = \sum_{j=1}^{\infty} \pi_j(x) e_j \). If \( T \in [l(p_i)] \), then \( Tx = \sum_{j=1}^{\infty} \pi_j(Tx) e_j \). Note that \( \pi_i(Tx) = \pi_i(\sum_{j=1}^{\infty} \pi_j(x) Te_j) = \sum_{j=1}^{\infty} \pi_i(Te_j) \pi_j(x) \), for \( i \in \mathbb{N}_+ \), by the continuity of \( \pi_i \). Consequently, if \( a_{ij} = \pi_i(Te_j) \), then \( Tx = \sum_{j=1}^{\infty} \pi_i(Tx) e_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \pi_j(x) e_i \). Therefore, for \( T \in [l(p_i)] \), a double sequence \( [a_{ij}] \subset \mathbb{C} \) will be called the matrix of \( T \) with respect to \( \{e_i\} \) (or simply the matrix of \( T \)) if and only if \( a_{ij} = \pi_i(Te_j) \).

Define \( \mathcal{C}[l(p_i)] \subset [l(p_i)] \) as follows: \( T \in \mathcal{C}[l(p_i)] \) if and only if \( T \in [l(p_i)] \) and for \( j \in \mathbb{N}_+ \), there exists \( n \in \mathbb{N}_+ \), depending on \( j \), such that \( Te_j \in E_n \). If \( T \in \mathcal{C}[l(p_i)] \) and \( [a_{ij}] \) is the matrix of \( T \), then there exists \( n \in \mathbb{N}_+ \) such that \( a_{ij} = 0 \) for \( n < i \). Also, \( T, U \in \mathcal{C}[l(p_i)] \) implies \( TU \in \mathcal{C}[l(p_i)] \). For \( T \in \mathcal{C}[l(p_i)] \), \( T \) is said to be almost superdiagonal (a.sd.) if and only if \( Te_j \in E_{j+1} \) for each \( j \in \mathbb{N}_+ \).

For \( T \in [l(p_i)], n \in \mathbb{N}_+ \), and \( [a_{ij}] \), the matrix of \( T \), let the matrix of \( T^n \) be denoted by \( [a_{ij}^{(n)}] \). It can be shown that if \( T \) is almost superdiagonal, then

\[
(a_{j+n,j}^{(m)}) = 0 \quad \text{for } m < n,
\]
\[
a_{j+n,j}^{(n)} = \prod_{i=0}^{n-1} a_{j+i,j+i} \quad \text{for } j, m, n \in \mathbb{N}_+
\]  

(1.3)

(see [1, Section 3, Theorem 3.6]).

An asterisk appended to the upper-left corner of a symbol indicates the nonstandard extension of the object represented by the symbol. The notation
\(\mu(0)\) will denote the set of *infinitesimals* of \(*\mathbb{C}\). An element \(\lambda\) of \(*\mathbb{C}\) is said to be *finite* if and only if \(|\lambda| \leq ^*\delta\) for some positive \(\delta \in \mathbb{R}\); otherwise, \(\lambda\) is said to be *infinite*. It is customary to consider \(\mathbb{C} \subset *\mathbb{C}\), that is, the elements of \(\mathbb{C}\) are identified with their nonstandard extensions; therefore, the asterisk notation is mostly not used for these elements. However, to bring clarity to some arguments, the asterisk notation for nonstandard extensions of elements of \(\mathbb{C}\) will be used occasionally.

The notation \(\mathcal{N}_\tau(0)\) denotes the \(\tau(p_i)\)-neighborhood filter of zero in \(l(p_i)\) and \(\mu_\tau(0)\) denotes the *monad* of the \(\tau(p_i)\)-neighborhoods of zero in \(l(p_i)\). An element \(\lambda\) of \(*l(p_i)\), denoted by \(^*\lambda\), such that \(\lambda \in \mu_\tau(0)\). Also, for \(A \subset *l(p_i)\), the set \(^*A \subset l(p_i)\), called the *standard part* of \(A\), is defined as follows: \(x \in ^*A\) if and only if there is a (unique) \(z \in A\) such that \(z - ^*x \in \mu_\tau(0)\). It can be shown that if \(F\) is an *internal* vector subspace of \(*l(p_i)\), then \(^*F\) is a \(\tau(p_i)\)-closed linear subspace of \(l(p_i)\) (see [3, Proposition 1.7]). If \(T \in [l(p_i)]\), then \(^*T[\mu_\tau(0)] \subseteq \mu_\tau(0)\), \(^*T(z)\) is near standard if \(z \in l(p_i)\) is near standard, and \(^*[^*T(z)] = T(z)\). We will denote the (external) set of all near standard points of \(*l(p_i)\) by the notation \(\mathcal{N}_\tau(*l(p_i))\). The (external) set \(\{z \in *l(p_i) \mid \lambda z \in \mu_\tau(0)\text{ for each } \lambda \in \mu(0)\}\) is called the set of *finite points* of \(*l(p_i)\) and is denoted by \(\mathcal{F}_\tau(*l(p_i))\). Clearly,

\[
\mathcal{N}_\tau(*l(p_i)) \subset \mathcal{F}_\tau(*l(p_i)). \tag{1.4}
\]

Finally, if \(Y\) is any set belonging to the *superstructure* generated by \(\mathbb{C} \cup l(p_i)\), then \(\triangle(Y)\) denotes the collection of all finite subsets of the set \(Y\). Also, the elements of \(\triangle(Y)\) are called *finite* subsets of \(*Y*.

### 2. Nonstandard properties of \(l(p_i)\).

The purpose of this section is to state some of the (nonstandard) properties of \((l(p_i), \tau(p_i))\) that will be used in later arguments. These facts were developed in [2, 3]. The reader is referred to these references for the proofs.

Recall that for \(x \in l(p_i)\),

\[
P_i(x) = \sum_{j=1}^i \pi_j(x)e_j \in E_i, \tag{2.1}
\]

where \(\{e_i\}\) is the natural Schauder basis for \(l(p_i)\), \(E_i = \text{sp}(e_1, \ldots, e_i)\), and \(\{\pi_i\}\) is the sequence of scalar projections generated by \(\{e_i\}\). Let \(\mathcal{F}(l(p_i))\) be the collection of all finite-dimensional linear subspaces of \(l(p_i)\). We will let \(d : \mathcal{F}(l(p_i)) \rightarrow \mathbb{N}\) denote the dimension function. In other words, \(d(F) = n\), for \(F \in \mathcal{F}(l(p_i))\), if and only if \(F = \text{sp}(x_1, \ldots, x_n)\) for some linearly independent \(\{x_i\}_{i=1}^n \subset l(p_i)\). In particular,

\[
d(E_i) = i \quad \text{for each } i \in \mathbb{N}_+. \tag{2.2}
\]
PROPOSITION 2.1 (see [2, Propositions III.1 and III.3]). If \( \alpha \in \mathbb{N}_+ - \mathbb{N}_+ \), then the (internal) projection \( P_\alpha : *l(p_1) \rightarrow E_\alpha \) satisfies the following two conditions:

1. for \( W \in N_\tau(0) \), there exists \( V \in N_\tau(0) \) such that \( P_\alpha[*V] \subset *W \);
2. if \( x \in l(p_i) \), then \( P_\alpha(*x) - *x \in \mu_\tau(0) \) (i.e., \( [*P_\alpha(*x)] = *x \)).

Note that Proposition 2.1 implies that for \( x \in l(p_i) \) and \( \alpha \in \mathbb{N}_+ - \mathbb{N}_+ \), we have \( l(p_i) = *E_\alpha \) and \( P_\alpha[\mu_\tau(0)] \subseteq \mu_\tau(0) \). It can be shown that

\[
z \in ns_\tau(*l(p_1)) \text{ implies } z - P_\alpha(z) \in \mu_\tau(0) \tag{2.3}
\]

(see [2, Proposition II.2]). Also, for \( T \in [l(p_i)] \) and \( \alpha \in \mathbb{N}_+ - \mathbb{N}_+ \), if we define \( T_\alpha = P_\alpha(*T)P_\alpha \), then \( T_\alpha \in *\mathcal{F}l(p_1) \) (i.e., \( T_\alpha \) is an internal linear transformation on \( *l(p_1) \)), \( T_\alpha : *l(p_i) \rightarrow E_\alpha \), and \( T_\alpha[\mu_\tau(0)] \subseteq \mu_\tau(0) \). In addition, \( [*T_\alpha(*x)] = T(x) \) for \( x \in l(p_i) \) and \( [*T_\alpha(z)] = T(z) \) for any near standard \( z \in *l(p_i) \) (see [2, Propositions II.4 and II.5]). Finally, if \( F \in *\mathcal{F}l(p_1) \) such that \( F \subseteq E_\alpha \) and \( T_\alpha[F] \subseteq F \), then \( T[F] \subseteq F \) (see [2, Proposition II.6]).

PROPOSITION 2.2 (see [2, Theorem II.1]). There exists a function \( \nabla : *\mathcal{F}l(p_1) \rightarrow \mathcal{F}[l(p_1)] \) that satisfies the following conditions:

1. if \( F \in \mathcal{F}l(p_1) \), then \( \nabla(F) : l(p_i) \rightarrow F \);
2. for each \( V \in N_\tau(0) \) and any nonzero \( x \in l(p_i) \), there exists a positive \( \lambda \in \mathbb{R} \) such that \( \nabla(F)(\lambda x) \in V \) for all \( F \in \mathcal{F}l(p_i) \);
3. if \( x \in l(p_i) \) such that \( x \in *F \) for \( F \in *\mathcal{F}l(p_1) \), then \( *\nabla(F)(*x) - *x \in \mu_\tau(0) \) (i.e., \( [*\nabla(F)(*x)] = *x \)).

Let \( [\mathcal{F}l(p_1)] \) be the collection of all linear transformations \( Q \) with \( \mathcal{D}(Q) \), \( \mathcal{R}(Q) \in \mathcal{F}l(p_1) \), where \( \mathcal{D}(Q) \) and \( \mathcal{R}(Q) \) are the domain and range of \( Q \), respectively, (i.e., the domain and range of linear transformation \( Q \) are finite dimensional for \( Q \in [\mathcal{F}l(p_1)] \)). Since the scalar field of \( l(p_i) \) is complex, the following sentence is true.

If \( E \in \mathcal{F}l(p_1) \) and \( Q \in [\mathcal{F}l(p_1)] \) such that \( Q : E \rightarrow E \), then for \( n = d(E) \), there exists \( \{F_j\}_{j=0}^n \in \triangle(\mathcal{F}l(p_1)) \) such that

(a) \( F_0 = \{0\} \) and \( F_n = E \),
(b) \( F_{j-1} \subset F_j \) for \( j = 1, \ldots, n \),
(c) \( d(F_j) = d(F_{j-1}) + 1 \) for \( j = 1, \ldots, n \),
(d) \( Q[F_j] \subset F_j \) for \( j = 0, \ldots, n \).

Note that the (logical) constants of the previous statement are \( \mathcal{F}l(p_1) \), \( [\mathcal{F}l(p_1)] \), \( \triangle(\mathcal{F}l(p_1)) \), and \( d \), the dimension function. Therefore, by the transfer principle, the following sentence is true.

If \( E \in *\mathcal{F}l(p_1) \) and \( Q \in *\mathcal{F}l(p_1) \) such that \( Q : E \rightarrow E \), then for \( \alpha = *d(E) \), there exists \( \{F_i\}_{i=0}^\alpha \in *\triangle(\mathcal{F}l(p_1)) \) such that

(a) \( F_0 = \{0\} \) and \( F_\alpha = E \),
(b) \( F_{i-1} \subset F_i \) for \( i = 1, \ldots, \alpha \),
(c) \( *d(F_i) = *d(F_{i-1}) + 1 \) for \( i = 1, \ldots, \alpha \),
(d) \( Q[F_i] \subset F_i \) for \( i = 0, \ldots, \alpha \).
In [2], this fact was used to obtain the following proposition.

**Proposition 2.3** (see [2, Definition II.2 and Lemma II.8]). Let \( \nabla : \mathcal{F}(l(p_i)) \to \mathcal{F}[l(p_i)] \) be the function established by **Proposition 2.2** and let \( \alpha \in \ast \mathbb{N}_+ - \mathbb{N}_+ \). If \( T \in \mathcal{L}(l(p_i)) \), then there exists an internal family \( \{F_i\}_{i=0}^\alpha \in \ast \Delta(\mathcal{F}(l(p_i))) \) such that the following conditions are fulfilled:

1. \( F_0 = \{0\}, F_\alpha = E_\alpha, \) and \( F_{i-1} \subset F_i \) for \( i = 1, \ldots, \alpha \).
2. \( \ast d(F_i) = \ast d(F_{i-1}) + 1 \) for \( i = 1, \ldots, \alpha \).
3. \( T_\alpha[F_i] \subset F_i \) for \( i = 0, \ldots, \alpha \), where \( T_\alpha = P_\alpha(\ast T)P_\alpha \).
4. \( \{F_i\}_{i=0}^\alpha \) and \( \{\ast \nabla(F_i)\}_{i=0}^\alpha \) are \( \ast \)-finite.
5. \( \ast \nabla(F_i) : \ast l(p_i) \to F_i \) such that \( x \in \ast F_i \) implies \( \ast \nabla(F_i)(\ast x) = \ast x \in \mu_\tau(0) \)
   (i.e., \( \ast \nabla(F_i)(\ast x) = x \) for each \( i = 0, \ldots, \alpha \).

Note that \( \ast E_\alpha = \ast l(p_i) = l(p_i) \) and from continuity and **Proposition 2.3**(3), we infer that \( T[l_F] \subset F_i \) for \( T \in [l(p_i)] \) and \( i = 0, \ldots, \alpha \). Also, given \( F_{i-1} \) and \( F_i \), \( i = 1, \ldots, \alpha \), it can be shown that **Proposition 2.3**(2) implies that for \( x_1, x_2 \in F_i \),

either \( x_1 = \zeta_1 x_2 + y_1 \) or \( x_2 = \zeta_2 x_1 + y_2 \) for some \( \zeta_1, \zeta_2 \in \mathbb{C} \) and \( y_1, y_2 \in \ast F_{i-1} \).

In other words, any two points of \( \ast F_i \) are linearly dependent modulo \( \ast F_{i-1} \) (see [2, Proposition I.21]).

Observe that for \( T \in [l(p_i)] \), **Proposition 2.3** produces a chain of closed invariant linear subspaces for \( T \), namely \( \{F_i\}_{i=0}^\alpha \). The problem is that we could have \( \ast F_i = \{0\} \) for \( i = 0, \ldots, \alpha \) \( \cap \mathbb{N} \) (i.e., the finite elements of \( \{0, \ldots, \alpha\} \)) and \( \ast F_i = l(p_i) \) for \( i = 0, \ldots, \alpha \) \( \cap \{\ast \mathbb{N} - \mathbb{N}\} \) (i.e., the infinite elements of \( \{0, \ldots, \alpha\} \)).

However, if we could find \( \nu \in \{1, \ldots, \alpha\} \) such that \( \ast F_{\nu-1} = l(p_i) \) and \( \ast F_\nu \neq \{0\} \), then either \( \ast F_{\nu-1} \) or \( \ast F_\nu \) is a closed nontrivial linear subspace of \( T \) since \( l(p_i) \) is infinite dimensional and any two points of \( \ast F_\nu \) are linearly dependent modulo \( \ast F_{\nu-1} \). The next proposition gives sufficient conditions for the existence of such a \( \nu \).

**Proposition 2.4** (see [2, Definition II.2 and Lemma II.9]). Let \( T \in \mathcal{L}(l(p_i)) \), \( \nabla : \mathcal{F}(l(p_i)) \to \mathcal{F}[l(p_i)] \) be the function established by **Proposition 2.2**, and let \( \alpha \in \ast \mathbb{N}_+ - \mathbb{N}_+ \). Let the collection \( \{F_i\}_{i=0}^\alpha : \{\ast \nabla(F_i)\}_{i=0}^\alpha \) satisfy the conditions of **Proposition 2.3** with respect to \( T \), \( \nabla \), and \( \alpha \). Let \( U \in \ast \mathcal{L}(l(p_i)) \) such that \( U[\mu_\tau(0)] \subset \mu_\tau(0) \). If there exists \( x \in l(p_i) \) such that \( U(\ast x) \not\in \mu_\tau(0) \) and \( U(\ast \nabla(F_i)(\ast x)) \in F_i \) nsr \( (\ast l(p_i)) \) for each \( i = 0, \ldots, \alpha \), then there exists \( \nu \in \{1, \ldots, \alpha\} \) such that \( \ast F_{\nu-1} = l(p_i) \) and \( \ast F_\nu \neq \{0\} \).

We close this section with a useful characterization of \( \text{fin}_\tau(\ast l(p_i)) \), the finite points of \( \ast l(p_i) \).

**Proposition 2.5.** If \( z \in \text{fin}_\tau(\ast l(p_i)) \), then \( \pi_i(z) \) is finite for \( \pi_i \in \ast \{\pi_i \mid i \in \mathbb{N}_+\} \).

**Proof.** Since \( 0 < p \leq \liminf p_i \) implies that \( \tau(p_i) \) is locally bounded (see [5, Theorem 6]), there exists a positive \( \delta_0 \in \mathbb{R} \) such that

\[
V_0 = S(\rho(p_i); \delta_0) = \{x \in l(p_i) \mid \rho(p_i)(x, 0) \leq \delta_0\} \tag{2.4}
\]
is \(\tau(p_i)\)-bounded. Let \(z \in \text{fin}_\tau(*l(p_i))\). Hence, \(*g_{V_0}(z) \leq *\delta\) for some positive \(\delta \in \mathbb{R}\), where \(g_{V_0}\) is the gauge of \(V_0\) (see [2, Proposition I.14]). So, \(z \in *\{(\delta V_0) \subset *S(\rho(p_i); \lambda)\) for \(\lambda = \max(\delta \delta_0, \delta_0, 1)\) since \(V_0\) is closed, balanced and \(\delta S(\rho(p_i); \delta_0) \subset S(\rho(p_i); \lambda)\). Therefore, \(*\rho(p_i)(z, 0) \leq *\lambda\), which implies \(|\pi_i(z)|^{\rho_i} \leq *\lambda\) for \(\pi_i \in *\{\pi_i | i \in \mathbb{N}_+\}\). It suffices to consider the case when \(1 \leq |\pi_i(z)|\) for \(i \in *\mathbb{N}_+\). Since \(p \in \mathbb{R}\) and \(0 < p_i^{-1} \leq *\lambda^{-1}\) for \(p_i \in *\{p_i\}\), we have \(|\pi_i(z)| \leq (*\lambda)\rho_i^{-1} \leq *(\lambda^{-1})\) for \(\pi_i \in *\{\pi_i | i \in \mathbb{N}_+\}\). We infer that \(\pi_i(z)\) is finite for \(\pi_i \in *\{\pi_i | i \in \mathbb{N}_+\}\).

### 3. Polynomials of compact almost superdiagonal operators.

We want to show that if \(T \in \mathbb{C}\) is almost superdiagonal and \(q(T)\) is compact for some polynomial \(q(\lambda)\) over \(\mathbb{C}\), then \(T\) has a nontrivial closed invariant linear subspace. Note that for \(\alpha \in *\mathbb{N}_+ - \mathbb{N}_+\) and \(\nabla : \mathbb{CP}(l(p_i)) \to \mathbb{CP}[l(p_i)]\), the function defined by Proposition 2.2, we can use Proposition 2.3 to produce a collection \([l(p_i)]\}_{i=0}^\alpha \subset *\{(\nabla F_i)^{\alpha}_{\nabla=\alpha}\} \subset \mathbb{CP}(l(p_i)\), for some \(\alpha \in *\mathbb{N}_+\) such that \([l(p_i)]\}_{i=0}^\alpha\) is a collection of closed invariant linear subspaces of \(T\). The strategy is to find some \(\alpha \in *\mathbb{N}_+ - \mathbb{N}_+\) such that \(*\rho(T_\alpha) \in *\mathbb{CP}(l(p_i))\) satisfies the hypotheses of Proposition 2.4, where \(T_\alpha = P_\alpha(T)\).

First, however, consider a compact operator \(U\) on \(l(p_i)\).

As stated in the proof of Proposition 2.5, there exists a positive \(\delta_0 \in \mathbb{R}\) such that

\[
V_0 = S(\rho(p_i); \delta_0) = \{x \in l(p_i) | \rho(p_i)(x, 0) \leq \delta_0\} \quad (3.1)
\]

is \(\tau(p_i)\)-bounded since \(0 < p \leq \liminf p_i\) implies that \(\tau(p_i)\) is locally bounded (see the known facts about \((l(p_i), \tau(p_i))\) in the first section). Therefore,

\[
*V_0 \subset \text{fin}_\tau(*l(p_i)) \quad (3.2)
\]

(see [2, Corollary I.18]). If \(U \in [l(p_i)]\) such that \(U[l(p_i)]\) is \(\tau(p_i)\)-compact for some \(W \in \mathcal{K}(0)\), then \(U[l(V_0)]\) is \(\tau(p_i)\)-compact since \(\lambda V_0 \subset W\) for some positive scalar \(\lambda\). Unless stated otherwise, \(V_0 = S(\rho(p_i); \delta_0)\) will be a fixed, \(\tau(p_i)\)-bounded neighborhood of \(0\), with \(0 < \delta_0 \leq 1\). Thus, we have that \(U \in [l(p_i)]\) is compact if and only if \(U[l(V_0)]\) is \(\tau(p_i)\)-compact. Also, if \(U[l(V_0)]\) is \(\tau(p_i)\)-compact, then \(*U[*V_0] \subset *\mathcal{N}_*(l(p_i))\) (see [2, Proposition I.11]).

**Proposition 3.1.** If \(U \in [l(p_i)]\) is compact and \([b_{ij}]\) is the matrix of \(U\), then \(b_{ik} \in \mu(0)\) for \(b_{ik} \in *[b_{ij}]\) such that \(i, k \in *\mathbb{N}_+ - \mathbb{N}_+\).

**Proof.** Let \(\lambda_k = (*\delta_0)^{p_k^{-1}}\) for \(k \in *\mathbb{N}_+\) and \(p_k \in *\{p_i\}\) and define \(z_k = \lambda_k e_k\) for \(k \in *\mathbb{N}_+\) and \(e_k \in *\{e_i\}\). We infer that \(*\rho(p_i)(z_k, 0) = |\lambda_k|^{p_k} \leq *\delta_0\) for \(k \in *\mathbb{N}_+\), which implies \(z_k \in *V_0\) for \(k \in *\mathbb{N}_+\). Therefore, \(*Uz_k \in *\mathcal{N}_*(l(p_i))\) for \(k \in *\mathbb{N}_+\) since \(*U[*V_0] \subset *\mathcal{N}_*(l(p_i))\). So, if \(t \in *\mathbb{N}_+ - \mathbb{N}_+\), then \(*Uz_k - P_{t-1}(Uz_k) \in \mu(0)\) for \(k \in *\mathbb{N}_+\) by expression (2.3) since \(t \in *\mathbb{N}_+ - \mathbb{N}_+\) implies \(t-1 \in *\mathbb{N}_+ - \mathbb{N}_+\).
Let \( \iota, \kappa \in \mathbb{N}_+ - \mathbb{N}_+ \) and let \([b_{ij}]\) be the matrix of \( U \) with respect to \( \{e_i\} \). Note that \( \pi_i(*Uz) = \lambda_k \pi_i(\pi U e_i) = \lambda_k b_{ik} \) for \( \pi_i \in *\pi \{ e_i \} \), \( e_i \in *\pi \{ e_i \} \), and \( b_{ik} \in *\pi \{ b_{ij} \} \). Since

\[
|\lambda_k b_{ik}|^{p_i} = \ast \rho(p_i)(\lambda_k b_{ik} e_i, 0) \leq \ast \rho(p_i)(\ast U z - P_{t-1}(*U z), 0),
\]

we have \( |\lambda_k b_{ik}|^{p_i} \in \mu(0) \). Consequently, \( 1 \leq p_i^{-1} \leq \ast(p^{-1}) \) implies \( \lambda_k |b_{ik}| \in \mu(0) \). Also, \( 1 \leq p_k^{-1} \leq \ast(p^{-1}) \) implies \( \ast(\delta_0^{p_k-1}) \leq \lambda_k \leq \ast\delta_0 \), which implies \( |b_{ik}| \in \mu(0) \). Therefore, \( b_{ik} \in \mu(0) \).

**Proposition 3.2.** If \( U \in \l[I(p_i)\r] \) is almost superdiagonal and \( q(U) \) is compact for some complex polynomial \( q(\lambda) \), then there exists \( \alpha \in \mathbb{N}_+ - \mathbb{N}_+ \) such that \( a_{\alpha+1,\alpha} \in \mu(0) \) for \( a_{\alpha+1,\alpha} \in *\pi \{ a_{ij} \} \), where \( \pi \{ a_{ij} \} \) is the matrix of \( U \).

**Proof.** Let \( n \) be the degree of \( q(\lambda) = \sum_{k=0}^{n} c_k \lambda^k \), which implies \( c_n \neq 0 \). If \( [b_{ij}] \) is the matrix of \( q(U) \), with respect to \( \{e_i\} \), then \( q(U) \) being compact implies \( b_{ik} \in \mu(0) \) for \( b_{ik} \in *\pi \{ b_{ij} \} \) such that \( \iota, \kappa \in \mathbb{N}_+ - \mathbb{N}_+ \) by Proposition 3.1. Let \( k \in \mathbb{N}_+ - \mathbb{N}_+ \). Since \( U \) being almost superdiagonal implies \( a_{k+n,k}^{(m)} = 0 \) for \( m < n \) and \( a_{k+n,k}^{(n)} = \prod_{l=0}^{n-1} a_{k+i+1,k+i} \) (by expression (1.3) and the transfer principle), we have \( b_{k+n,k} = \ast c_n \prod_{l=0}^{n-1} a_{k+i+1,k+i+1} \). Therefore, if \( k \in \mathbb{N}_+ - \mathbb{N}_+ \), then \( c_n \notin \mu(0) \) and \( b_{k+n,k} \in \mu(0) \) imply \( a_{k+i+1,\kappa+i} \in \mu(0) \) for some \( i_0 \in \{0, \ldots, n-1\} \). Let \( \alpha = k+i_0 \).

**Proposition 3.3.** If \( U \in \mathcal{L}(l(p_i)) \) such that \( U[\mu(0)] \subset \mu(0) \), then

\[
U[\text{fin} \ast l(p_i)] \subset \text{fin} \ast l(p_i). \tag{3.4}
\]

**Proof.** Let \( z \in \text{fin} \ast l(p_i) \). If \( \lambda \in \mu(0) \), then \( \lambda z \in \mu(0) \), which implies \( \lambda U z = U(\lambda z) \in \mu(0) \). Therefore, \( U z \in \text{fin} \ast l(p_i) \).

Let \( T \in \mathcal{F}[l(p_i)] \) be an almost superdiagonal operator such that \( q(T) \) is compact for \( q(\lambda) = \sum_{k=0}^{n} c_k \lambda^k \), a polynomial over \( \mathbb{C} \) with \( c_n \neq 0 \). Let \( \nabla : \mathcal{F}[l(p_i)] \to \mathcal{F}[l(p_i)] \) satisfy the conditions of Proposition 2.2 and let \( \alpha \in *\mathbb{N}_+ - \mathbb{N}_+ \) satisfy the conclusion of Proposition 3.2. Note that the (internal) projection \( P_{\alpha} : \ast l(p_i) - E_{\alpha} \) satisfies the conditions of Proposition 2.1. Define \( T_{\alpha} = P_{\alpha}(\ast T)P_{\alpha} \). Observe that

\[
* q(T_{\alpha}) \in *\mathcal{L}(l(p_i)), \quad T_{\alpha} \subset \mu(0), \quad * q(T) \subset \mu(0),
\]

since \( T \) and \( q(T) \) are continuous.

**Proposition 3.4.** If \( z \in \mu(0) \), then \( * q(T_{\alpha}) z \in \mu(0) \).

**Proof.** It suffices to show that \( (T_{\alpha})^m \mu(0) \subset \mu(0) \) for \( m \in \mathbb{N}_+ \) (see [2, Proposition I.6]). Note that from (3.5), \( T_{\alpha} \mu(0) \subset \mu(0) \). Assume that
\[(T_\alpha)^m[\mu_\tau(0)] \subset \mu_\tau(0) \text{ for } m \in \mathbb{N}_+. \text{ Consequently,} \]

\[(T_\alpha)^{m+1}[\mu_\tau(0)] = T_\alpha[(T_\alpha)^m[\mu_\tau(0)]] \subset T_\alpha[\mu_\tau(0)] \subset \mu_\tau(0). \quad (3.6)\]

Therefore, \((T_\alpha)^m[\mu_\tau(0)] \subset \mu_\tau(0)\) for each \(m \in \mathbb{N}_+\) by induction. \(\square\)

So, one of the conditions of Proposition 2.4 for \(\ast q(T_\alpha)\) has been satisfied, that is, \(\ast q(T_\alpha)[\mu_\tau(0)] \subset \mu_\tau(0)\).

**Proposition 3.5.** Let \(F \in \mathcal{F}(l(p_1))\) such that \(F \subset E_\alpha\). If \(T_\alpha[F] \subset F\), then \((T_\alpha)^m[F] \subset F\) for \(m \in \mathbb{N}\).

**Proof.** If \((T_\alpha)^m[F] \subset F\) for \(m \in \mathbb{N}\), then \((T_\alpha)^{m+1}[F] = T_\alpha[(T_\alpha)^m[F]] \subset T_\alpha[F] \subset F\). Therefore, \((T_\alpha)^m[F] \subset F\) for any \(m \in \mathbb{N}\) by induction. \(\square\)

Consequently, if \(F \in \mathcal{F}(l(p_1))\) such that \(F \subset E_\alpha\) and \(T_\alpha[F] \subset F\), then

\[\ast q(T_\alpha)[F] \subset F. \quad (3.7)\]

**Proposition 3.6.** If \(z \in E_\alpha \cap \text{fin}_\tau(\ast l(p_1))\), then \([\ast q(T_\alpha)z - \ast (q(T))z] \subset \mu_\tau(0)\).

**Proof.** It is sufficient to show that \(\ast(T^m)z - (T_\alpha)^mz \subset \mu_\tau(0)\) for \(z \in E_\alpha \cap \text{fin}_\tau(\ast l(p_1))\) and \(m \in \mathbb{N}\) (see \([2, \text{Proposition I.6}]\)). Let \(z \in E_\alpha \cap \text{fin}_\tau(\ast l(p_1))\), which implies \(z = \sum_{k=1}^{\alpha} \pi_k(z)e_k\) for \(\pi_k \in \ast \{\pi_i \mid i \in \mathbb{N}_+\}\) and \(e_k \in \ast \{e_i\}\). Consequently,

\[\ast Tz = \sum_{k=1}^{\alpha} \pi_k(z) \ast T e_k = \sum_{k=1}^{\alpha} \pi_k(z) \left[ \sum_{i=1}^{\alpha+1} a_{ik} e_i \right] = \sum_{i=1}^{\alpha+1} \left[ \sum_{k=1}^{\alpha} a_{ik} \pi_k(z) \right] e_i, \quad (3.8)\]

since \(T\) is almost superdiagonal. Also, \(a_{\alpha+1,k} = 0\) for \(k < \alpha\), which implies \(\sum_{k=1}^{\alpha} a_{\alpha+1,k} \pi_k(z) = a_{\alpha+1,\alpha} \pi_\alpha(z)\). Therefore, \(\ast Tz = \sum_{i=1}^{\alpha} [\sum_{k=1}^{\alpha} a_{ik} \pi_k(z)] e_i + a_{\alpha+1,\alpha} \pi_\alpha(z) e_{\alpha+1}\), which implies \(\ast Tz = P_\alpha(\ast Tz) = a_{\alpha+1,\alpha} \pi_\alpha(z) e_{\alpha+1}\). So,

\[\ast \rho(p_1)(\ast Tz - P_\alpha(\ast Tz), 0) = \left| a_{\alpha+1,\alpha} \right|^{P_{\alpha+1}} \left| \pi_\alpha(z) \right|^{P_{\alpha+1}}. \quad (3.9)\]

Since \(\pi_\alpha(z)\) is finite (by Proposition 2.5), \(a_{\alpha+1,\alpha} \in \mu(0)\) (Proposition 3.2), and \(0 < \ast p \leq P_{\alpha+1} \leq 1\), we infer that \(\left| a_{\alpha+1,\alpha} \right|^{P_{\alpha+1}} \left| \pi_\alpha(z) \right|^{P_{\alpha+1}} \in \mu(0)\), which implies \(\ast Tz - P_\alpha(\ast Tz) \in \mu_\tau(0)\). Therefore, \(\ast Tz - T_\alpha z \in \mu_\tau(0)\) since \(z \in E_\alpha\) implies \(z = P_\alpha(z)\).

Now, let \(m \in \mathbb{N}\) such that \(2 \leq m\) and assume that \(\ast(T^{m-1})z - (T_\alpha)^{m-1}z \in \mu_\tau(0)\) for \(z \in E_\alpha \cap \text{fin}_\tau(\ast l(p_1))\). If \(z \in E_\alpha \cap \text{fin}_\tau(\ast l(p_1))\), then

\[\ast(T^m)z - \ast T((T_\alpha)^{m-1}z) = \ast T(\ast(T^{m-1})z - (T_\alpha)^{m-1}z) \in \mu_\tau(0). \quad (3.10)\]
since \( T \in [I(p_1)] \) implies that \( T \) is linear and \( *T[\mu_\tau(0)] \subseteq \mu_\tau(0) \). If we set \( y = (T_\alpha)^{m-1}z \), then \( y \in E_\alpha \cap \text{fin}_\tau(*I(p_1)) \) by Propositions 3.3 and 3.5 since \( T_\alpha[E_\alpha] \subseteq E_\alpha \) and \( (T_\alpha)^{m-1}[\mu_\tau(0)] \subseteq \mu_\tau(0) \) (see the proof of Proposition 3.4). Thus,

\[
*T \left( (T_\alpha)^{m-1}z \right) - (T_\alpha)^m z = *T(y) - T_\alpha y \in \mu_\tau(0)
\]

(see the first part of the present proof), which implies

\[
*(T^m)z - (T_\alpha)^m z = \left[ *(T^m)z - *T \left( (T_\alpha)^{m-1}z \right) \right]
+ \left[ *T \left( (T_\alpha)^{m-1}z \right) - (T_\alpha)^m z \right] \in \mu_\tau(0).
\]

Therefore, by induction, it follows that \( *(T^m)z - (T_\alpha)^m z \in \mu_\tau(0) \) for \( z \in E_\alpha \cap \text{fin}_\tau(*I(p_1)) \) and \( m \in \mathbb{N} \).

**PROPOSITION 3.7.** If \( z \in E_\alpha \cap \text{fin}_\tau(*I(p_1)) \), then \( *q(T_\alpha)z \in \text{ns}_\tau(*I(p_1)) \) (i.e., \( *q(T_\alpha)z \) is \( \tau(p_i)\)-near standard).

**PROOF.** Let \( z \in E_\alpha \cap \text{fin}_\tau(*I(p_1)) \). There exists \( n \in \mathbb{N} \) such that \( z \in *\left( nV_0 \right) \) (see [2, Corollary I.15]). Since \( q(T) \) is compact, it follows that

\[
*(q(T))[*(nV_0)] = n*(q(T))[V_0] \subseteq n[\text{ns}_\tau(*I(p_1))] \subset \text{ns}_\tau(*I(p_1))
\]

(see [2, Proposition I.1 and Corollary I.10]). Therefore,

\[
*q(T_\alpha)z - *[q(T)]z = \left[ *q(T_\alpha)z - *(q(T))z \right]
+ \left[ *(q(T))z - *[q(T)]z \right] \in \mu_\tau(0)
\]

(3.14)

since \( *[q(T_\alpha)z - *(q(T))z] \in \mu_\tau(0) \) by Proposition 3.6. Therefore, \( *q(T_\alpha)z \in \text{ns}_\tau(*I(p_1)) \).

We now state and prove the main result.

**THEOREM 3.8.** Let \( 0 < p \leq p_i \leq 1 \) and let \( T \in [l(p_i)] \) be almost superdiagonal. If \( q(\lambda) \) is a polynomial over \( \mathbb{C} \) such that \( q(T) \) is compact, then \( T \) has at least one nontrivial \( \tau(p_i)\)-closed invariant linear subspace of \( l(p_i) \).

**PROOF.** Let \( [a_{ij}] \) be the matrix of \( T \) with respect to \( \{e_i\} \). Therefore, there exists \( \alpha \in *N_+ \setminus N_+ \) such that \( a_{\alpha+1,\alpha} = \mu(0) \) for \( a_{\alpha+1,\alpha} \in *[a_{ij}] \) by Proposition 3.2. Let \( \nabla : \mathcal{F}(l(p_i)) \to \mathcal{F}(l(p_i)) \) satisfy the conditions of Proposition 2.2 and let the collection \( \{F_{i_\alpha} \}_{i_\alpha}^{\alpha} : \{*_\nabla(F_i)\}_{i_\alpha}^{\alpha} \) satisfy the conclusion of Proposition 2.3 with respect to \( T, \nabla, \) and \( \alpha \). From Proposition 2.2(2) (and the transfer principle), we infer the existence of a nonzero \( x_0 \in l(p_i) \) such that \( _*\nabla(F)(*_x_0) \in *V_0 \) for each \( F \in *\mathcal{F}(l(p_i)) \), which implies \( _*\nabla(F)(*_x_0) \in F \cap \text{fin}_\tau(*I(p_1)) \) for each
$F \in \mathcal{T}^{f}(I(p_{1}))$ (see expression (3.2)). Consequently,

$$^{\ast}q(T_{\alpha})({}^{\ast}\nabla(F_{i})({}^{\ast}x_{0})) \in F_{i} \cap \text{ns}_{\tau}(^{\ast}l(p_{1})) \quad \text{for } i \in \{0, \ldots, \alpha\}$$

(3.15)

by Proposition 3.7 since, for each $i \in \{0, \ldots, \alpha\}$, $^{\ast}\nabla(F_{i})({}^{\ast}x_{0}) \in F_{i} \subset E_{\alpha}$, by definitions of $\nabla$, $\{F_{i}\}_{i=0}^{\alpha}$ (and the transfer principle), and $^{\ast}q(T_{\alpha})[F_{i}] \subset F_{i}$ (see expression (3.7)).

If $\{x_{0}, Tx_{0}, \ldots, T^{m}x_{0}\}$ is linearly dependent for some $m \in \mathbb{N}_{+}$, then the linear space generated by $\{x_{0}, Tx_{0}, \ldots, T^{m}x_{0}\}$ is nontrivial, closed, and invariant under $T$.

For the remainder of the proof, we will assume that $\{x_{0}, Tx_{0}, \ldots, T^{m}x_{0}\}$ is linearly independent for each $m \in \mathbb{N}_{+}$. Consequently,

$$q(T)x_{0} \neq 0.$$  

(3.16)

Since the (internal) projection $P_{\alpha}$ satisfies Proposition 2.1, we have $^{\ast}x_{0} - P_{\alpha}(^{\ast}x_{0}) \in \mu_{\tau}(0)$, which implies $[{^{\ast}q(T_{\alpha})}(^{\ast}x_{0}) - ^{\ast}q(T_{\alpha})(P_{\alpha}(^{\ast}x_{0}))] \in \mu_{\tau}(0)$ by Proposition 3.4 and $[{^{\ast}q(T)}(P_{\alpha}(^{\ast}x_{0})) - ^{\ast}(q(T)(x_{0}))] \in \mu_{\tau}(0)$ because $q(T) \in [I(p_{1})]$. Also, $P_{\alpha}(^{\ast}x_{0}) \in E_{\alpha} \cap \text{fin}_{\tau}(^{\ast}l(p_{1}))$ (see expression (1.4)) implies

$$[{^{\ast}q(T_{\alpha})}(P_{\alpha}(^{\ast}x_{0})) - ^{\ast}(q(T)(P_{\alpha}(^{\ast}x_{0})))] \in \mu_{\tau}(0)$$

(3.17)

by Proposition 3.6. Therefore,

$$^{\ast}q(T_{\alpha})(^{\ast}x_{0}) - ^{\ast}(q(T)(x_{0}))$$

$$= [{^{\ast}q(T_{\alpha})}(^{\ast}x_{0}) - ^{\ast}q(T_{\alpha})(P_{\alpha}(^{\ast}x_{0}))]$$

$$+ [{^{\ast}q(T_{\alpha})}(P_{\alpha}(^{\ast}x_{0})) - ^{\ast}(q(T)(^{\ast}x_{0}))]$$

$$+ [{^{\ast}q(T)}(P_{\alpha}(^{\ast}x_{0})) - ^{\ast}(q(T)(x_{0}))],$$

(3.18)

which implies $[{^{\ast}q(T_{\alpha})}(^{\ast}x_{0}) - ^{\ast}(q(T)(x_{0}))] \in \mu_{\tau}(0)$. So, $q(T)x_{0} \neq 0$ implies

$$^{\ast}q(T_{\alpha})(^{\ast}x_{0}) \notin \mu_{\tau}(0)$$

(3.19)

since $\tau = \tau(p_{1})$ is Hausdorff. Therefore, by Propositions 2.4 and 3.4, expressions (3.15) and (3.19), there exists $\nu \in \{1, \ldots, \alpha\}$ such that $^{\ast}F_{\nu-1} \neq I(p_{1})$ and $^{\ast}F_{\nu} \neq \{0\}$. Since any two points of $^{\ast}F_{\nu}$ are linearly dependent modulo $^{\ast}F_{\nu-1}$, we have that either $^{\ast}F_{\nu-1}$ or $^{\ast}F_{\nu}$ is a closed nontrivial linear subspace of $T$.

\[\square\]

References


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