COMPOSITION OPERATORS FROM THE BLOCH SPACE INTO THE SPACES $Q_T$

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Suppose that $\varphi(z)$ is an analytic self-map of the unit disk $\Delta$. We consider the boundedness of the composition operator $C_\varphi$ from Bloch space $\mathcal{B}$ into the spaces $Q_T (Q_{T,0})$ defined by a nonnegative, nondecreasing function $T(r)$ on $0 \leq r < \infty$.

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1. Introduction. Let $\Delta = \{z : |z| < 1\}$ be the unit disk of complex plane $\mathbb{C}$ and let $H(\Delta)$ be the space of all analytic functions in $\Delta$. For $a \in \Delta$, Green’s function with logarithmic singularity at $a \in \Delta$ is denoted by $g(z,a) = \log |(1-\bar{a}z)/(a-z)|$. For $0 < p < \infty$, the space $Q_p$ consists of all functions $f$ analytic in $\Delta$ for which

$$\sup_{a \in \Delta} \left( \int_{\Delta} |f'(z)|^2 (g(z,a))^p \, dA(z) \right)^{1/p} < \infty,$$  

(1.1)

where $dA(z)$ is the Euclidean area element on $\Delta$.

$Q_p$-spaces have been investigated by many authors (cf. [1, 2, 3, 9]). We know that $Q_1 = \text{BMOA}$, the space of all analytic functions of bounded mean oscillation (cf. [4]). Further, the spaces $Q_p$ are the same for each $p \in (1, \infty)$, and each space equals to the Bloch space $\mathcal{B}$, which is a Banach space with the norm

$$\|f\|_\mathcal{B} := |f(0)| + \|f\|_b := |f(0)| + \sup_{z \in \Delta} (1 - |z|^2) |f'(z)|.$$  

(1.2)

Recently, we introduced a new space $Q_T$ (cf. [5, 10]) by a nondecreasing function $T(r)$ on $0 \leq r < \infty$ as follows.

**Definition 1.1.** Let $T(r) \not\equiv 0$ be a nonnegative, nondecreasing function on $0 \leq r < \infty$. A function $f \in H(\Delta)$ is said to belong to $Q_T$ if

$$\|f\|_{Q_T}^2 := \sup_{a \in \Delta} \left( \int_{\Delta} |f'(z)|^2 T(g(z,a)) \, dA(z) \right) < \infty.$$  

(1.3)

If

$$\lim_{|a|\to 1} \left( \int_{\Delta} |f'(z)|^2 T(g(z,a)) \, dA(z) \right) = 0,$$  

(1.4)

then $f$ is said to belong to $Q_{T,0}$. 
For $0 < p < \infty$, if we take $T(r) = r^p$, the space $Q_T$ coincides with the space $Q_p$. We note that $Q_T \subset B$ for all nondecreasing functions $T$. We have previously shown that $Q_T = Q_p$ under certain growth conditions on $T(r)$ (cf. [10]).

In the present paper, first we give some basic properties of $Q_T$ spaces, some of which are also new for the special case $Q_T = Q_p$. For example, $Q_T$ is a Banach space with the norm defined by

$$\|f\|_{Q_T} := |f(0)| + \|f\|_{Q_T}.$$  \hspace{1cm} (1.5)

Then we investigate the boundedness of the composition operators from the Bloch space $B$ into $Q_T$ or $Q_{T,0}$. These results extend some previously known results (cf. [6, 8]).

2. Basic properties of $Q_T$ spaces. We give the following propositions.

**Proposition 2.1.** The space $Q_T$ is a subspace of the Bloch space $B$.

The proof of Proposition 2.1 can be found in [10].

**Proposition 2.2.** The space $Q_T$ is a Banach space with the norm defined in (1.5).

**Proof.** For $f \in Q_T$ and $a \in \Delta$, define

$$I^2(f, a) := \iint_{\Delta} |f'(z)|^2 T(g(z, a)) dA(z).$$ \hspace{1cm} (2.1)

Let $f_1, f_2 \in Q_T$. It follows from Schwarz’s inequality that

$$\iint_{\Delta} |f_1(z)f_2(z)| T(g(z, a)) dA(z) \leq I(f_1, a)I(f_2, a),$$ \hspace{1cm} (2.2)

and then

$$I^2(f_1 + f_2, a) \leq I^2(f_1, a) + 2I(f_1, a)I(f_2, a) + I^2(f_2, a)$$

$$= (I(f_1, a) + I(f_2, a))^2.$$ \hspace{1cm} (2.3)

Thus, $I(f_1 + f_2, a) \leq I(f_1, a) + I(f_2, a)$ for all $a \in \Delta$. Hence

$$\|f_1 + f_2\|_{Q_T} \leq \|f_1\|_{Q_T} + \|f_2\|_{Q_T}.$$ \hspace{1cm} (2.4)

Therefore,

$$\|f_1 + f_2\|^2_{Q_T} = \left( |f_1(0) + f_2(0)| + \|f_1 + f_2\|_{Q_T} \right)^2$$

$$\leq \left( |f_1(0)| + |f_2(0)| + \|f_1\|_{Q_T} + \|f_2\|_{Q_T} \right)^2$$

$$= \left( \|f_1\|_{Q_T} + \|f_2\|_{Q_T} \right)^2.$$ \hspace{1cm} (2.5)
that is, \( \|f_1 + f_2\|_T \leq \|f_1\|_T + \|f_2\|_T \). On the other hand, it is obvious that \( \|f\|_T \geq 0 \) for each \( f \in Q_T \) and that \( \|f\|_T = 0 \) if and only if \( f \equiv 0 \). It is obvious that \( \|c f\|_T = |c| \|f\|_T \) for any constant \( c \). Thus, \( Q_T \) is a normed space.

Let \( f \in Q_T \) and let \( \phi_a(w) = (a - w)/(1 - \bar{a}w) \), \( a \in \Delta \). Then by changing a variable \( w = \phi_a(z) \), we obtain

\[
\|f\|_T^2 \geq \int_{\Delta} \left| f'(z) \right|^2 T(g(z, a)) dA(z)
= \int_{\Delta} \left| (f \circ \phi_a)'(w) \right|^2 T\left( \frac{1}{|w|} \right) dA(w)
\geq T\left( \log \frac{1}{r} \right) \int_{|w| < r} \left| (f \circ \phi_a)'(w) \right|^2 dA(w)
\geq \pi r^2 T\left( \log \frac{1}{r} \right) (1 - |a|^2)^2 \left| f'(a) \right|^2.
\]

(2.6)

For \( r_0, 0 < r_0 < 1 \), such that \( T(\log(1/r_0)) \neq 0 \), we have

\[
\|f\|_b \leq \frac{\|f\|_T^2}{r_0(\pi T(\log 1/r_0))^{1/2}}.
\]

(2.7)

Since \( f \in Q_T \subset \mathcal{H} \), we have for \( z \in \Delta \),

\[
\left| f(z) \right| \leq \left| f(0) \right| + \frac{\|f\|_b}{2} \log \frac{1 + |z|}{1 - |z|}
\leq \left| f(0) \right| + \frac{\|f\|_T^2}{2r_0(\pi T(\log 1/r_0))^{1/2}} \log \frac{1 + |z|}{1 - |z|}
\leq \|f\|_T \left( 1 + \frac{1}{2r_0(\pi T(\log 1/r_0))^{1/2}} \right) \left( \frac{1 + |z|}{1 - |z|} \right).
\]

(2.8)

Suppose \( \{f_n\} \) is a Cauchy sequence in \( Q_T \). Then there is a constant \( M > 0 \) such that

\[
\|f_n\|_T \leq M, \quad n = 1, 2, \ldots.
\]

(2.9)

By the estimate (2.8) for a fixed \( r_0 \in (0, 1) \), we obtain that

\[
\left| f_n(z) \right| \leq M \left( 1 + \frac{1}{2r_0(\pi T(\log 1/r_0))^{1/2}} \right) \left( \frac{1 + |z|}{1 - |z|} \right).
\]

(2.10)
compact subsets of $\Delta$ by inequality (2.10). By Fatou’s lemma, we get that
\[
\begin{aligned}
\int \int_{\Delta} |f'(z)|^2 T(g(z,a)) \, dA(z) \\
= \int \int_{\Delta} \lim_{j \to \infty} |f_{n_j}'(z)|^2 T(g(z,a)) \, dA(z) \\
\leq \liminf_{j \to \infty} \int \int_{\Delta} |f_{n_j}'(z)|^2 T(g(z,a)) \, dA(z) \\
\leq \liminf_{j \to \infty} \|f_{n_j}\|_{Q_T}^2 \leq M^2 
\end{aligned}
\] (2.11)
holds for all $a \in \Delta$, so that $f \in Q_T$. By a similar reasoning, we can prove that \( \|f_n - f\|_T \to 0 \) as $n \to \infty$. The proof of Proposition 2.2 is complete.

3. Boundedness of composition operators. Let $\varphi(z)$ be an analytic self-map of the unit disk $\Delta$. Let the composition operator $C_\varphi$ induced by $\varphi$ from $H(\Delta)$ to itself be defined by $C_\varphi(f) = f \circ \varphi$ for $f \in H(\Delta)$. The boundedness of composition operators from $\mathcal{B}$ to itself and from $\mathcal{B}$ to $Q_p$ have been studied in [6, 8], respectively. In this paper, we consider the same problems for the general spaces $Q_T$.

**Theorem 3.1.** Let $T(r) \not\equiv 0$ be a nonnegative, nondecreasing function on $0 \leq r < \infty$ and let $\varphi$ be an analytic self-map of $\Delta$. Then $C_\varphi : \mathcal{B} \to Q_T$ is bounded if and only if
\[
\sup_{a \in \Delta} \int \int_{\Delta} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} T(g(z,a)) \, dA(z) < \infty. 
\] (3.1)

**Proof.** Let (3.1) hold and let $K_1(K_1 > 0)$ be the supremum in (3.1). If $f \in \mathcal{B}$, then for all $a \in \Delta$, we have
\[
\begin{aligned}
\int \int_{\Delta} |(C_\varphi f)'(z)|^2 T(g(z,a)) \, dA(z) \\
= \int \int_{\Delta} |f'(\varphi(z))| |\varphi'(z)|^2 T(g(z,a)) \, dA(z) \\
\leq \|f\|_{\mathcal{B}}^2 \int \int_{\Delta} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} T(g(z,a)) \, dA(z) \\
\leq K_1^2 \|f\|_{\mathcal{B}}^2.
\end{aligned}
\] (3.2)
Consequently, $\|C_\varphi f\|_{Q_T} \leq K_1 \|f\|_{\mathcal{B}}$. Since $f(z) \in \mathcal{B}$, we obtain
\[
\|C_\varphi f\|_{T}^2 = \left( |f \circ \varphi(0)| + \|C_\varphi f\|_{Q_T} \right)^2 \\
\leq \left( |f(0)| + \frac{\|f\|_{\mathcal{B}}}{2} \log \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} + K_1 \|f\|_{\mathcal{B}} \right)^2 \\
\leq K^2 \left( |f(0)| + \|f\|_{\mathcal{B}} \right)^2 = K^2 \|f\|_{\mathcal{B}}^2, 
\] (3.3)
where $K = \max\{1, K_1 + (1/2) \log(1 + |\varphi(0)|) / (1 - |\varphi(0)|)\}$. Thus, $\|C_{\varphi} f\|_T \leq K \|f\|_b$, which shows that $C_{\varphi} : B \to Q_T$ is bounded.

Conversely, assume that $C_{\varphi} : B \to Q_T$ is bounded, there exists a constant $K > 0$ such that for each $f \in B$, we have

$$\|C_{\varphi} f\|_T \leq K \|f\|_b.$$  \hfill (3.4)

On the other hand, by a result in [7], there exist $f_1, f_2 \in B$ such that

$$\frac{1}{1 - |z|^2} \leq |f_1'(z)| + |f_2'(z)|$$ \hfill (3.5)

holds for all $z \in \Delta$, so that

$$\frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} \leq 2 \left( |f_1 \circ \varphi)'(z)|^2 + |f_2 \circ \varphi)'(z)|^2 \right).$$ \hfill (3.6)

Thus, the inequalities

$$\iint_{\Delta} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} T(g(z,a)) dA(z) \leq 2 \iint_{\Delta} \left( |f_1 \circ \varphi)'(z)|^2 + |f_2 \circ \varphi)'(z)|^2 \right) T(g(z,a)) dA(z) \leq 2K^2 \left( \|f_1\|_{\mathcal{B}}^2 + \|f_2\|_{\mathcal{B}}^2 \right)$$ \hfill (3.7)

hold for all $z, a \in \Delta$, which establishes (3.1). The proof of Theorem 3.1 is completed. \hfill \square

**Remark 3.2.** Note that if $C_{\varphi} : B \to B$, then (3.1) holds for any increasing function $T$ satisfying $Q_T = B$. Indeed, we know that $Q_T = B$ (see [5]) if and only if

$$\int_{0}^{1} T\left(\log\left(\frac{1}{r}\right)\right) (1 - r^2)^{-2} r dr < \infty.$$ \hfill (3.8)

The Schwarz-Pick lemma guarantees that $((1 - |z|^2) / (1 - |\varphi(z)|^2)) |\varphi'(z)| \leq 1$, so that (3.8) leads easily to (3.1). It means that $C_{\varphi} : B \to B$ is always bounded (cf. [6]).

**Remark 3.3.** If one considers the composition operator $C_{\varphi}$ from the Bloch space to the Dirichlet space

$$\mathcal{D} = \left\{ f \in H(\Delta) : \iint_{\Delta} |f'(z)|^2 dA(z) < \infty \right\},$$ \hfill (3.9)

then $C_{\varphi} : B \to \mathcal{D}$ is bounded if and only if

$$\iint_{\Delta} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} dA(z) < \infty.$$ \hfill (3.10)
For the spaces $Q_{T,0}$, we have the following results.

**Theorem 3.4.** Let $T(r)$ be a nonnegative, nondecreasing function on $0 \leq r < \infty$ and let $\varphi$ be an analytic self-map of $\Delta$. Then $C_\varphi : \mathbb{B} \to Q_{T,0}$ is bounded if and only if

$$
\lim_{|a| \to 1} \int_\Delta \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} T(g(z,a))dA(z) = 0. \quad (3.11)
$$

**Proof.** Suppose $C_\varphi : \mathbb{B} \to Q_{T,0}$ is bounded. Using a way similar to the proof of Theorem 3.1, we choose functions $f_1, f_2 \in \mathbb{B}$ such that

$$
\frac{1}{1 - |z|^2} \leq |f_1(z)| + |f_2(z)| \quad (3.12)
$$

for all $z \in \Delta$. Then $C_\varphi f_1$ and $C_\varphi f_2$ belong to $Q_{T,0}$. Therefore,

$$
\lim_{|a| \to 1} \int_\Delta \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} T(g(z,a))dA(z)
\leq 2 \lim_{|a| \to 1} \int_\Delta \left( |(f_1 \circ \varphi)'(z)|^2 + |(f_2 \circ \varphi)'(z)|^2 \right) T(g(z,a))dA(z) = 0, \quad (3.13)
$$

which shows that (3.11) holds.

Conversely, by Theorem 3.1, we know that $C_\varphi : \mathbb{B} \to Q_T$ is bounded since condition (3.11) implies that

$$
\sup_{a \in \Delta} \int_\Delta \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} T(g(z,a))dA(z) < \infty. \quad (3.14)
$$

We need only to prove that $C_\varphi f \in Q_{T,0}$ for each $f \in \mathbb{B}$, and this follows from the inequality

$$
\int_\Delta |(C_\varphi f)'(z)|^2 T(g(z,a))dA(z)
= \int_\Delta |f'(\varphi(z))|^2 |\varphi'(z)|^2 T(g(z,a))dA(z)
\leq \|f\|_p^2 \int_\Delta \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} T(g(z,a))dA(z). \quad (3.15)
$$

The proof of Theorem 3.4 is completed.

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