ON THE CONVOLUTION PRODUCT OF THE DISTRIBUTIONAL KERNEL $K_{\alpha,\beta,\gamma,\nu}$

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We introduce a distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ which is related to the operator $\oplus^k$ iterated $k$ times and defined by

$$\oplus^k = \left[ \left( \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left( \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k,$$

where $p + q = n$ is the dimension of the space $\mathbb{R}^n$ of the $n$-dimensional Euclidean space, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $k$ is a nonnegative integer, and $\alpha$, $\beta$, $\gamma$, and $\nu$ are complex parameters. It is found that the existence of the convolution $K_{\alpha,\beta,\gamma,\nu} \ast K_{\alpha',\beta',\gamma',\nu'}$ is depending on the conditions of $p$ and $q$.

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1. Introduction. The operator $\oplus^k$ can be factorized in the form

$$\oplus^k = \left[ \left( \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left( \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \times \left[ \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \left[ \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k,$$

where $p + q = n$ is the dimension of the space $\mathbb{R}^n$, $i = \sqrt{-1}$, and $k$ is a nonnegative integer. The operator $(\sum_{r=1}^p \partial^2/\partial x_r^2)^2 - (\sum_{j=p+1}^{p+q} \partial^2/\partial x_j^2)^2$ is first introduced by Kananthai [2] and named the Diamond operator denoted by

$$\diamond = \left( \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left( \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2.$$

We denote the operators $L_1$ and $L_2$ by

$$L_1 = \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2},$$

$$L_2 = \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}.$$
Thus (1.1) can be written by
\[ \oplus^k = \Diamond^k L_1^k L_2^k. \] (1.4)

Now consider the convolution \( RH_\alpha(u) \ast R_\beta^e(v) \ast S_y(w) \ast T_\nu(z) \) where \( RH_\alpha(u) \), \( R_\beta^e(v) \), \( S_y(w) \), and \( T_\nu(z) \) are defined by (2.2), (2.4), (2.6), and (2.7), respectively.

We defined the distributional kernel \( K_{\alpha,\beta,y,\nu} \) by
\[ K_{\alpha,\beta,y,\nu} = RH_\alpha(u) \ast R_\beta^e(v) \ast S_y(w) \ast T_\nu(z). \] (1.5)

Since the functions \( RH_\alpha(u) \), \( R_\beta^e(v) \), \( S_y(w) \), and \( T_\nu(z) \) are all tempered distributions and the supports of \( RH_\alpha(u) \) and \( R_\beta^e(v) \) are compact (see [2, pages 30–31] and [1, pages 152–153]), then the convolution on the right-hand side of (1.5) exists and also is a tempered distributions. Thus \( K_{\alpha,\beta,y,\nu} \) is well defined and also is a tempered distribution.

For \( \alpha = \beta = \gamma = \nu = 2k \), we obtain \((-1)^k K_{2k,2k,2k,2k}\) as an elementary solution of the operator \( \oplus^k \), see [3]. That is \( \oplus^k (-1)^k K_{2k,2k,2k,2k}(x) = \delta \) where \( \delta \) is the Dirac-delta distribution and \( \oplus^k \) is defined by (1.4).

2. Preliminaries

**Definition 2.1.** Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and write
\[ x = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \quad p + q = n. \] (2.1)

Denote by \( \Gamma_+ = \{ x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0 \} \) the interior of forward cone and \( \Gamma_+ \) denote its closure. For any complex number \( \alpha \), we define the function
\[ RH_\alpha(x) = \begin{cases} \frac{u^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+, \\ 0, & \text{if } x \notin \Gamma_+, \end{cases} \] (2.2)

where the constant \( K_n(\alpha) \) is given by the formula
\[ K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma((2 + \alpha - n)/2) \Gamma((1 - \alpha)/2) \Gamma(\alpha)}{\Gamma((2 + \alpha - p)/2) \Gamma((p - \alpha)/2)}, \] (2.3)

the function \( RH_\alpha(x) \) is first introduced by Nozaki [4, page 72] and is called the ultra-hyperbolic kernel of Marcel Riesz. Hence \( RH_\alpha(x) \) is an ordinary function if \( \text{Re}(\alpha) \geq n \) and is a distribution of \( \alpha \) if \( \text{Re}(\alpha) < n \). Let \( \text{supp} RH_\alpha(u) \subset \Gamma_+ \) where \( \text{supp} RH_\alpha(u) \) denotes the support of \( RH_\alpha(u) \).
ON THE CONVOLUTION PRODUCT OF THE DISTRIBUTIONAL KERNEL

**Definition 2.2.** Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and write \( v = x_1^2 + x_2^2 + \cdots + x_n^2 \).

For any complex number \( \beta \), define the function
\[
R_{\beta}^e(v) = \frac{v(\beta - n)/2}{W_n(\beta)},
\]
where \( W_n(\beta) = \pi^{n/2} \beta \Gamma(\beta)/\Gamma((n - \beta)/2) \), the function \( R_{\beta}^e(v) \) is called the elliptic kernel of Marcel Riesz and is ordinary function if \( \text{Re}(\beta) \geq n \) and is a distribution of \( \beta \) if \( \text{Re}(\beta) < n \).

**Definition 2.3.** Let \( x = (x_1, x_2, \ldots, x_n) \) be a point of the space \( \mathbb{R}^n \). Write
\[
w = x_1^2 + x_2^2 + \cdots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2),
\]
\[
z = x_1^2 + x_2^2 + \cdots + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2), \quad p + q = n, \ i = \sqrt{-1}.
\]

For any complex numbers \( \gamma \) and \( \nu \), define
\[
S_\gamma(\omega) = \frac{\omega(\gamma - n)/2}{W_n(\gamma)},
\]
\[
T_\nu(z) = \frac{z(\nu - n)/2}{W_n(\nu)},
\]
where \( W_n(\gamma) = \pi^{n/2} \gamma \Gamma(\gamma/2)/\Gamma((n - \gamma)/2) \), \( W_n(\nu) = \pi^{n/2} \nu \Gamma(\nu/2)/\Gamma((n - \nu)/2) \).

**Lemma 2.4** (the convolution product of \( R_{\beta}^e(v) \)). The convolution \( R_{\beta}^e * R_{\beta'}^e = R_{\beta + \beta'}^e \) where \( R_{\beta}^e \) and \( R_{\beta'}^e \) are given by (2.2).

**Proof.** See [5, page 20].

**Lemma 2.5** (the convolution product of \( R_{\alpha}^H(x) \)). The convolution product is given by

(i)
\[
R_{\alpha}^H * R_{\alpha'}^H = \frac{\cos (\alpha(\pi/2)) \cos (\alpha'(\pi/2))}{\cos ((\alpha + \beta)/2)\pi} \cdot R_{\alpha + \alpha'}^H,
\]

where \( R_{\alpha}^H \) and \( R_{\alpha'}^H \) are defined by (2.1) with \( p \) even,
(ii) $R_{\alpha}^H \ast R_{\alpha}^H = R_{\alpha+\alpha'}^H + A_{\alpha,\alpha'}$ for $p$ odd, where

$$A_{\alpha,\alpha'} = \frac{2\pi i}{4} \frac{C((-\alpha - \alpha')/2)}{C(-\alpha/2)C(-\alpha'/2)} \left[ H_{\alpha+\alpha'}^+ - H_{\alpha+\alpha'}^- \right],$$

$$C(r) = \Gamma(r)\Gamma(1-r),$$

$$H_{\gamma}^\pm(x \pm i0, n) = e^{\pm r(\pi/2)i} e^{\pm q(\pi/2)i} a\left(\frac{r}{2}\right)(u \pm i0)^{(r-n)/2},$$

$$u = u(x) \text{ is defined by (2.1)} \text{ and } |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} \text{ in particular } R_{\alpha}^H \ast R_{-2k}^H = R_{\alpha-2k}^H \text{ and } R_{\alpha}^H \ast R_{-2k}^H = R_{\alpha+2k}^H.$$

The proof of this lemma is given by Téllez [6, pages 121–123].

**Lemma 2.6** (the convolutions product of $S_{\gamma}(w)$ and $T_{\nu}(z)$). The convolutions product is given by

(i) $S_{\gamma} \ast S_{\gamma'} = (i)^{q/2} S_{\gamma+y'}$,

(ii) $T_{\nu} \ast T_{\nu'} = (-i)^{q/2} T_{\nu+y'}$ where $S_{\gamma}$ and $T_{\nu}$ are defined by (2.6) and (2.7), respectively.

**Proof.** (i) Now

$$\langle S_{\gamma}(w), \varphi(x) \rangle = \frac{1}{W_n(y)} \int_{\mathbb{R}^n} \omega^{(y-n)/2} \varphi(x) dx,$$  \quad (2.11)

where $\varphi \in \mathcal{S}$ the space of infinitely differentiable function with compact supports. We have $\omega = x_1^2 + x_2^2 + \cdots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2)$, $p + q = n$. By changing the variables $x_1 = y_1, x_2 = y_2, \ldots, x_p = y_p, x_{p+1} = y_{p+1}/\sqrt{-l}, x_{p+2} = y_{p+2}/\sqrt{-l}, \ldots$, and $x_{p+q} = y_{p+q}/\sqrt{-l}$. Thus we obtain $\omega = y_1^2 + y_2^2 + \cdots + y_p^2 + y_{p+1}^2 + y_{p+2}^2 + \cdots + y_{p+q}^2$. Let $r^2 = y_1^2 + y_2^2 + \cdots + y_p^2$, $p + q = n$.

Thus (2.11) can be written in the form

$$\langle S_{\gamma}(w), \varphi(x) \rangle = \frac{1}{W_n(y)} \int_{\mathbb{R}^n} r^{y-n} \varphi \frac{\partial}{\partial (y_1, y_2, \ldots, y_n)} dy_1 dy_2 \cdots dy_n$$

$$= \frac{1}{(-i)^{q/2} W_n(y)} \int_{\mathbb{R}^n} r^{y-n} \varphi dy$$

$$= \frac{(i)^{q/2}}{W_n(y)} \langle r^{y-n}, \varphi \rangle.$$  \quad (2.12)

Thus $S_{\gamma}(w) = ((i)^{q/2}/w_n(y)) r^{y-n} = (i)^{q/2} R_{\gamma}(w)$ by (2.4).
Consider the convolution \( S_\gamma * S_{\gamma'} \). We have

\[
S_\gamma * S_{\gamma'} = (i)q/2 \Re \gamma(w) * (i)q/2 \Re \gamma'(w) = (i)q \Re \gamma + \gamma'(w)
\]

by Lemma 2.4 and [1, pages 157-159]

\[
= (i)q/2 \Re \gamma + \gamma'(w) = (i)q/2 S_\gamma + \gamma'(w).
\]

(2.13)

Similarly, for (ii) we also have

\[
T_\nu * T_{\nu'} = (-i)q/2 T_\nu + \nu'.
\]

(2.14)

3. Main results

**Theorem 3.1.** Let \( K_{\alpha,\beta,\gamma,\nu} \) be the distributional kernel defined by (1.5). Then we obtain the following:

(i) \( K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'} \) for \( \alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma', \) and \( \nu' \) positive even numbers with \( \alpha = \beta = \gamma = \nu, \alpha' = \beta' = \gamma' = \nu' \);

(ii) \( K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = B_{\alpha,\alpha'} \cdot K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'} \) for \( p \) even, \( \alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma', \) and \( \nu' \) any complex numbers, \( B_{\alpha,\alpha'} = \cos(\alpha \pi/2) \cos(\alpha' \pi/2) / \cos((\alpha + \alpha')/2) \pi \); 

(iii) \( K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'} + A_{\alpha,\alpha'} * R_{\beta+\beta'} * S_{\gamma+\gamma'} * T_{\nu+\nu'} \) for \( p \) odd, \( \alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma', \) and \( \nu' \) any complex numbers \( R_{\beta}, S_{\gamma}, \) and \( T_{\nu} \) defined by (2.4), (2.6), and (2.7), respectively. And

\[
A_{\alpha,\alpha'} = \frac{C((-\alpha - \alpha')/2)}{C(-\alpha/2)} \frac{2\pi i}{4} \left[ H_{\alpha+\alpha'}^+ - H_{\alpha+\alpha'}^- \right],
\]

\[
C(r) = \Gamma(r) \Gamma(1-r),
\]

\[
H_r^{\pm} = H_r(u \pm i0, n) = e^{\mp r(\pi/2)i} e^{\pm q(\pi/2)i} a\left(\frac{r}{2}\right)(u \pm i0)^{r-n/2},
\]

\[
a\left(\frac{r}{2}\right) = \Gamma\left(\frac{n-r}{2}\right) 2^{r-n/2} \pi^{n/2} \Gamma\left(\frac{r}{2}\right)^{-1},
\]

\[
(u \pm i0)^{\lambda} = \lim_{\epsilon \to 0} (u + i \in |x|^2)^{\lambda},
\]

where \( u = u(x) \) is defined by (2.1) and

\[
|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}.
\]

(3.1)

Proof. The proof of (i) follows from [3, Theorem 3.1, page 66]. The proof of (ii) and (iii) is obtained by Lemmas 2.4, 2.5, and 2.6.
REFERENCES


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