A RELATIVE INTEGRAL BASIS OVER $\mathbb{Q}(\sqrt{-3})$
FOR THE NORMAL CLOSURE
OF A PURE CUBIC FIELD

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Let $K$ be a pure cubic field. Let $L$ be the normal closure of $K$. A relative integral basis (RIB) for $L$ over $\mathbb{Q}(\sqrt{-3})$ is given. This RIB simplifies and completes the one given by Haghighi (1986).

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1. Introduction. Let $K$ be the pure cubic field $\mathbb{Q}(d^{1/3})$, where $d$ is a cube-free integer, and let $L$ be the normal closure of $K$ so that $\mathbb{Q} \subset K \subset L$, $[L : K] = 2$, and $[K : \mathbb{Q}] = 3$. Let $k$ be the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$ so that $\mathbb{Q} \subset k \subset L$, $[L : k] = 3$, and $[k : \mathbb{Q}] = 2$. The ring of all algebraic integers is denoted by $\Omega$. The rings of integers of $K, k, L$ are $\mathcal{O}_K = K \cap \Omega$, $\mathcal{O}_k = k \cap \Omega$, $\mathcal{O}_L = L \cap \Omega$, respectively. As $\mathcal{O}_k$ is a principal ideal domain, $L/k$ possesses a relative integral basis (RIB) [3, Corollary 3, page 401]. Haghighi [2, Theorems 5.1, 5.3, 5.6] has given a RIB for $L/k$. However, Haghighi’s RIB for $L/k$ contains two difficulties. The first is that in certain cases the RIB makes use of an element of norm 3 in a pure cubic field, a quantity which is not easy to determine, see [2, Theorem 5.1]. The second problem is that the RIB is not completely general, see [2, Theorem 5.3]. In this note, we give a simple and completely general RIB for $L/k$.

2. Preliminary remarks. As $d$ is a cube-free integer, we can define integers $a$ and $b$ by

$$d = ab^2, \quad (a, b) = 1, \ a, b \text{ square-free.} \quad (2.1)$$

If $a^2 \not\equiv b^2 \pmod{9}$, an integral basis for $K$ is

$$\left\{1, (ab^2)^{1/3}, (a^2b)^{1/3}\right\}, \quad (2.2)$$

and if $a^2 \equiv b^2 \pmod{9}$, an integral basis is

$$\left\{1, (ab^2)^{1/3}, \frac{b + ab(ab^2)^{1/3} + (a^2b)^{1/3}}{3}\right\}. \quad (2.3)$$
These integral bases are due to Dedekind [1]. From (2.2) and (2.3), we deduce that the discriminant \(d(K)\) of \(K\) is given by
\[
d(K) = -3f^2, \tag{2.4}
\]
where
\[
f = \begin{cases} 
3ab, & \text{if } a^2 \not\equiv b^2 \pmod{9}, \\
ab, & \text{if } a^2 \equiv b^2 \pmod{9}.
\end{cases} \tag{2.5}
\]
The relative discriminant \(d(L/k)\) of \(L/k\) is given by
\[
d(L/k) = f^2 = \begin{cases} 
9a^2b^2, & \text{if } a^2 \not\equiv b^2 \pmod{9}, \\
a^2b^2, & \text{if } a^2 \equiv b^2 \pmod{9},
\end{cases} \tag{2.6}
\]
see [1]. We note that if \(\alpha, \beta \in O_L\) are such that
\[
d_{L/k}(1, \alpha, \beta) = d(L/k), \tag{2.7}
\]
then \(\{1, \alpha, \beta\}\) is a RIB for \(L/k\).

3. RIB for \(L/k\). We show that \(\{1, \alpha, \beta\}\) is a RIB for \(L/k\), where \(\alpha\) and \(\beta\) are given in Table 3.1.

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>(\alpha)</th>
<th>(\beta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>3</td>
<td>a, 3</td>
<td>b</td>
</tr>
<tr>
<td>(ii)</td>
<td>3</td>
<td>a, 3</td>
<td>b</td>
</tr>
<tr>
<td>(iii)</td>
<td>3</td>
<td>a, 3</td>
<td>b, 9</td>
</tr>
<tr>
<td>(iv)</td>
<td>3</td>
<td>a, 3</td>
<td>b, 9</td>
</tr>
</tbody>
</table>

An easy calculation making use of (2.2), (2.3), (2.4), and (2.5) shows that
\[
d_{L/k}(1, \alpha, \beta) = \begin{cases} 
9a^2b^2, & \text{if } a^2 \not\equiv b^2 \pmod{9}, \\
a^2b^2, & \text{if } a^2 \equiv b^2 \pmod{9},
\end{cases} \tag{3.1}
\]
so that (2.7) holds in view of (2.6). Clearly, \(\alpha \in L\) and \(\beta \in L\). We now show that \(\alpha \in \Omega\) and \(\beta \in \Omega\) so that \(\alpha \in O_L\) and \(\beta \in O_L\), proving that \(\{1, \alpha, \beta\}\) is a RIB for \(L/k\). Clearly, \(\alpha \in \Omega\) in Cases (i) and (iii), and \(\beta \in \Omega\) in Cases (ii) and (iv), see (2.3)
for the latter. In the remaining cases, it suffices to give a monic polynomial $f_\alpha(x) \in \mathbb{Z}[x]$ of which $\alpha$ is a root in Cases (ii) and (iv), and a monic polynomial $f_\beta(x) \in \mathbb{Z}[x]$ of which $\beta$ is a root in Cases (i) and (iii).

**CASE (i).** Here,

$$f_\beta(x) = x^6 + 3a_1^4b^2, \quad a_1 = \frac{a}{3} \in \mathbb{Z}. \tag{3.2}$$

**CASE (ii).** Here,

$$f_\alpha(x) = x^6 + 3a_2^2b_1^4, \quad b_1 = \frac{b}{3} \in \mathbb{Z}. \tag{3.3}$$

**CASE (iii).** We have

$$a^2 \equiv b^2 \equiv 1 \pmod{3}, \quad a^2 - b^2 \equiv 0 \pmod{3}, \tag{3.4}$$

so that

$$a^4b^4 - 3a_2^2b^2 + a^2 + b^2 = (a^2 - b^2)^2 + (a^2 - 1)(b^2 - 1)(a^2b^2 + a^2 + b^2)$$

$$\equiv 0 \pmod{9}, \tag{3.5}$$

and we define $m \in \mathbb{Z}$ by

$$m = \frac{(a^4b^4 - 3a^2b^2 + a^2 + b^2)}{9}. \tag{3.6}$$

In this case,

$$f_\beta(x) = x^6 + (2a^2 + 1)b^2x^4 + ((a^2 - 1)^2b^2 - 6m)b^2x^2 + 3b^2m^2. \tag{3.7}$$

**CASE (iv).** We have

$$a^2 \equiv b^2 \equiv 1 \pmod{3}, \quad a^2 - b^2 \equiv 0 \pmod{9}, \quad a^2 + 2b^2 \equiv 0 \pmod{3} \tag{3.8}$$

so that we can define $r, s \in \mathbb{Z}$ by

$$r = \frac{(a^2 + 2b^2)}{3}, \quad s = \frac{(a^2 - b^2)}{9}. \tag{3.9}$$

Here,

$$f_\alpha(x) = x^6 + a^2x^4 + a^2rx^2 + 3a^2s^2. \tag{3.10}$$

This completes the proof that \{1, \alpha, \beta\} is a RIB for $L/k$.

We conclude with four examples.

**Example 3.1** (cf. [2, Illustration 5.2]). A RIB for $\mathbb{Q}(\sqrt[3]{213}, \sqrt{-3})$ over $\mathbb{Q}(\sqrt{-3})$ is (see Case (i))

$$\left\{1, 213^{1/3}, \frac{213^{2/3}}{\sqrt{-3}} \right\}. \tag{3.11}$$
Example 3.2. A RIB for $\mathbb{Q}(\sqrt[3]{3}, \sqrt{-3})$ over $\mathbb{Q}(\sqrt{-3})$ is (see Case (ii))

$$\left\{ 1, \frac{9^{1/3}}{\sqrt[3]{-3}}, 3^{1/3} \right\}.$$ (3.12)

Example 3.3 (cf. [2, Illustration 5.5]). A RIB for $\mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$ over $\mathbb{Q}(\sqrt{-3})$ is (see Case (iii))

$$\left\{ 1, 2^{1/3}, \frac{1 + 2 \cdot 2^{1/3} + 2^{2/3}}{\sqrt{-3}} \right\}.$$ (3.13)

Example 3.4 (cf. [2, Illustration 5.7]). A RIB for $\mathbb{Q}(\sqrt[3]{10}, \sqrt{-3})$ over $\mathbb{Q}(\sqrt{-3})$ is (see Case (iv))

$$\left\{ 1, \frac{10^{1/3} - 10}{\sqrt{-3}}, \frac{1 + 10 \cdot 10^{1/3} + 10^{2/3}}{3} \right\}.$$ (3.14)

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References


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