REMARKS ON EMBEDDABLE SEMIGROUPS IN GROUPS
AND A GENERALIZATION OF SOME CUTHBERT’S RESULTS

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Let \((U(t))_{t \geq 0}\) be a \(C_0\)-semigroup of bounded linear operators on a Banach space \(X\). In this paper, we establish that if, for some \(t_0 > 0\), \(U(t_0)\) is a Fredholm (resp., semi-Fredholm) operator, then \((U(t))_{t \geq 0}\) is a Fredholm (resp., semi-Fredholm) semigroup. Moreover, we give a necessary and sufficient condition guaranteeing that \((U(t))_{t \geq 0}\) can be embedded in a \(C_0\)-group on \(X\). Also we study semigroups which are near the identity in the sense that there exists \(t_0 > 0\) such that \(U(t_0) - I \in \mathcal{J}(X)\), where \(\mathcal{J}(X)\) is an arbitrary closed two-sided ideal contained in the set of Fredholm perturbations. We close this paper by discussing the case where \(\mathcal{J}(X)\) is replaced by some subsets of the set of polynomially compact perturbations.

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1. Introduction. Let \(X\) be a Banach space over the complex field and let \(\mathcal{L}(X)\) denote the Banach algebra of bounded linear operators on \(X\). The subset of all compact operators of \(\mathcal{L}(X)\) is designated by \(\mathcal{K}(X)\). For \(A \in \mathcal{L}(X)\), we let \(\sigma(A)\), \(\rho(A)\), \(R(\lambda, A)\), \(N(A)\), and \(R(A)\) denote the spectrum, the resolvent set, the resolvent operator, the null space, and the range of \(A\), respectively. The nullity of \(A\), \(\alpha(A)\), is defined as the dimension \(N(A)\) and the deficiency of \(A\), \(\beta(A)\), is defined as the codimension of \(R(A)\) in \(X\).

Write

\[
\Phi_+(X) = \{A \in \mathcal{L}(X) : \alpha(A) < \infty, \text{ R}(A) \text{ is closed in } X\},
\]

\[
\Phi_-(X) = \{A \in \mathcal{L}(X) : \beta(A) < \infty \text{ (then } R(A) \text{ is closed in } X\}\}. \tag{1.1}
\]

By \(\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)\) we denote the set of semi-Fredholm operators in \(\mathcal{L}(X)\), while \(\Phi(X) := \Phi_+(X) \cap \Phi_-(X)\) is the set of Fredholm operators in \(\mathcal{L}(X)\). If \(A \in \Phi_\pm(X)\), the number \(i(A) = \alpha(A) - \beta(A)\), a finite or infinite integer is the index of \(A\). Let \(X^*\) denotes the dual space of \(X\) and \(A^*\) the dual operator of \(A\).

Let \((U(t))_{t \geq 0}\) be a \(C_0\)-semigroup of bounded linear operators on \(X\). We say that \((U(t))_{t \geq 0}\) is a Fredholm (resp., semi-Fredholm) semigroup if \(U(t)\) is in \(\Phi(X)\) (resp., \(\Phi_\pm(X)\)) for all \(t > 0\).
In [7, Theorem 16.3.6], it is proved that a $C_0$-semigroup of bounded linear operators $(U(t))_{t \geq 0}$ can be embedded in a $C_0$-group if and only if there exists $t_0 > 0$ such that $0 \in \rho(U(t_0))$. The main goal of Section 2 is to give a generalization of this result to Fredholm semigroup. Our approach consists in relaxing the requirement there exists $t_0 > 0$ such that $0 \in \rho(U(t_0))$ and replacing it by the weaker one there exists $t_0 > 0$ such that $U(t_0) \in \Phi(X)$. In fact, we prove under this hypothesis that $(U(t))_{t \geq 0}$ is a Fredholm semigroup, that is, $U(t) \in \Phi(X)$ for all $t \geq 0$. In particular, we show that if there exists $t_0 > 0$ such that $U(t_0) \in \Phi_\pm(X)$, then $(U(t))_{t \geq 0}$ is a semi-Fredholm semigroup, that is, $U(t) \in \Phi_\pm(X)$ for all $t \geq 0$.

In Section 3, we extend some results owing to Cuthbert [2] which deal with $C_0$-semigroups having the property of being near the identity, in the sense that, for some value of $t$, $U(t) - I \in \mathcal{H}(X)$. We show that Cuthbert’s results remain valid if, for some $t_0 > 0$, $U(t_0) - I \in \mathcal{J}(X)$ where $\mathcal{J}(X)$ is an arbitrary closed two-sided ideal of $\mathcal{L}(X)$ contained in the ideal of Fredholm perturbations $\mathcal{F}(X)$. In the last section, some generalizations of the results obtained in Section 3 to polynomially compact perturbations are also given.

2. Embeddable $C_0$-semigroups in $C_0$-groups. Let $X$ be a Banach space and let $(U(t))_{t \geq 0}$ be a $C_0$-semigroup of bounded linear operators on $X$.

**Theorem 2.1.** A $C_0$-semigroup $(U(t))_{t \geq 0}$ can be embedded in a $C_0$-group on $X$ if and only if there exists $t_0 > 0$ such that $U(t_0) \in \Phi(X)$.

To prove Theorem 2.1, the following proposition is required.

**Proposition 2.2.** Let $t_0 > 0$ and let $(U(t))_{t \geq 0}$ be a $C_0$-semigroup on $X$.

(i) If $U(t_0) \in \Phi_+(X)$, then $U(t) \in \Phi_+(X)$ and $\alpha(U(t)) = 0$ for all $t \geq 0$.

(ii) If $U(t_0) \in \Phi_-(X)$, then $U(t) \in \Phi_-(X)$ and $\beta(U(t)) = 0$ for all $t \geq 0$.

(iii) If $U(t_0) \in \Phi(X)$, then $U(t) \in \Phi(X)$ and $i(U(t)) = 0$ for all $t \geq 0$.

Obviously, Proposition 2.2 shows that if, for some $t_0 > 0$, $U(t_0) \in \Phi_\pm(X)$, then $(U(t))_{t \geq 0}$ is a semi-Fredholm semigroup. In the case where $U(t_0) \in \Phi(X)$, $(U(t))_{t \geq 0}$ is a Fredholm semigroup and $i(U(t)) = 0$ for all $t \geq 0$.

**Proof of Proposition 2.2.** (i) We first show that $U(t_0)$ is injective. Since $\alpha(U(t_0)) < \infty$, then $0$ is an eigenvalue with finite multiplicity of $U(t_0)$. Let $x \neq 0$ be an eigenvector associated to $0$. Putting $t_1 = t_0/2$, then $U(t_0)x = U(t_1)U(t_1)x = 0$, hence $0$ is an eigenvalue of $U(t_1)$. Proceeding by induction, we define a sequence $(t_n)_{n \in \mathbb{N}}$ with $t_n \rightarrow 0$ as $n \rightarrow \infty$ such that $0$ is an eigenvalue of $U(t_n)$, $\forall n \in \mathbb{N}$. For $n \geq 0$, we define the sets

$$
\Lambda_n = N(U(t_n)) \cap \{x \in X : \|x\| = 1\}.
$$

(2.1)

Clearly, the inclusion $N(U(s)) \subseteq N(U(t))$, for $s \leq t$, and the compactness of $\Lambda_0$ imply that $(\Lambda_n)_{n}$ is a decreasing sequence (in the sense of the inclusion) of
nonempty compact subsets of $X$. Thus $\bigcap_{n=0}^{\infty} \Lambda_n \neq \emptyset$. If $x \in \bigcap_{n=0}^{\infty} \Lambda_n$, then
\[ ||U(t_n)x - x|| = ||x|| = 1 \quad \forall n \geq 1. \tag{2.2} \]

Since $t_n \to 0$ as $n \to \infty$, (2.2) contradicts the strong continuity of $(U(t))_{t \geq 0}$. This shows that $N(U(t)) = \{0\}$, that is, $\alpha(U(t_0)) = 0$.

Let $0 \leq t \leq t_0$. The inclusion $N(U(t)) \subseteq N(U(t_0))$ implies that $\alpha(U(t)) = 0$. Assume now that $t > t_0$ and $x \in N(U(t))$, then there exists an integer $n$ such that $nt_0 > t$ and therefore $U(nt_0)x = U(nt_0 - t)U(t)x = 0$. Hence, we have $x = 0$ and consequently $N(U(t)) = \{0\}$ for all $t > t_0$ which ends the proof of (i).

(ii) To prove this item, we will proceed by duality. Let $(U^*(t))_{t \geq 0}$ be the dual semigroup of $(U(t))_{t \geq 0}$. Since $\beta(U(t)) = \alpha(U^*(t))$, then it suffices to show that $\alpha(U^*(t)) = 0$ for all $t \geq 0$. By hypothesis, we have $\alpha(U^*(t_0)) < \infty$. Let $x^*$ be an element of $N(U^*(t_0))$. Arguing as above, we construct a sequence $(t_n)_{n \in \mathbb{N}}$ with $t_n \to 0$ as $n \to \infty$ such that $0$ is an eigenvalue of $U^*(t_n)$, for all $n \in \mathbb{N}$.

A decreasing sequence
\[ \Sigma_n = N(U^*(t_n)) \bigcap \{ x^* \in X^* : ||x^*|| = 1 \} \tag{2.3} \]
of nonempty compact subsets of $X^*$. We infer that $\bigcap_{n=0}^{\infty} \Sigma_n \neq \emptyset$. Let $x^* \in \bigcap_{n=0}^{\infty} \Sigma_n$, then for all $n \in \mathbb{N}$
\[ | \langle U^*(t_n)x^* - x^*, x \rangle | = | \langle x^*, x \rangle | \quad \forall x \in X. \tag{2.4} \]

Using the fact that $(U^*(t))_{t \geq 0}$ is continuous in the weak* topology at $t = 0$, we conclude that
\[ \lim_{t \to 0} | \langle U^*(t_n)x^* - x^*, x \rangle | = 0 \quad \forall x \in X. \tag{2.5} \]

Combining (2.4) and (2.5), we obtain $\langle x^*, x \rangle = 0$ for all $x \in X$. This shows that $x^* = 0$ and therefore $\alpha(U^*(t_0)) = 0$. Arguing as above, we show that $\alpha(U^*(t)) = 0$ for all $t \geq 0$.

(iii) This follows from (i) and (ii).

To complete the proof of (i) it suffices to show that $R(U(t))$ is closed in $X$ for all $t \geq 0$. Assume that $U(t_0) \in \Phi_+(X)$, then $\alpha(U(t_0)) < \infty$ and $\beta(U(t_0)) = \infty$ (if $\beta(U(t_0)) < \infty$ the proof is contained in (ii) see below). Let $U^*(t_0)$ be the dual operator of $U(t_0)$. Obviously, $U^*(t_0) \in \Phi_-(X)$ and consequently $\beta(U^*(t_0)) < \infty$. Hence $\beta(U^*(t)) < \infty$ for all $t \geq 0$. Now applying Kato’s lemma [8, Lemma 332] we infer that $R(U^*(t))$ is closed in $X^*$ for all $t \geq 0$. This together with the closed graph theorem of Banach [15, page 205] implies that $R(U(t))$ is closed in $X$ for all $t \geq 0$.

Assume now that $U(t_0) \in \Phi_-(X)$, then $\beta(U(t_0)) < \infty$ and $\alpha(U(t_0)) = \infty$ (if $\alpha(U(t_0)) < \infty$ the proof is contained in (i)). It follows from the first part of the statement (ii) that $\beta(U(t)) < \infty$ for all $t \geq 0$. Again using Kato’s lemma
we see that $R(U(t))$ is closed in $X$ for all $t \geq 0$ which completes the proof of (ii).

Now if $U(t_0) \in \Phi(X)$, then $\alpha(U(t_0)) < \infty$ and $\beta(U(t_0)) < \infty$. It follows from the discussion above that $R(U(t))$ is closed in $X$ for all $t \geq 0$. This ends the proof of Proposition 2.2.

**Proof of Theorem 2.1.** The proof follows immediately from Proposition 2.2 and [7, Theorem 16.3.6].

### 3. An extension of some results by Cuthbert

Throughout this section $X$ denotes a Banach space and $(U(t))_{t \geq 0}$ designates a strongly continuous semigroup with infinitesimal generator $A$.

As mentioned in the introduction, this section is motivated by Cuthbert’s work [2] dealing with $C_0$-semigroups which have the property of being near the identity, in the sense that, for some positive value of $t > 0$, $U(t) - I \in \mathcal{H}(X)$. We discuss the possibility of extending Cuthbert’s results to other operator ideals of $\mathcal{L}(X)$. To this purpose, we introduce the concept of Fredholm perturbations (see [1, 4, 12]).

**Definition 3.1.** We say that an operator $F \in \mathcal{L}(X)$ is a Fredholm perturbation if $A + F \in \Phi(X)$ whenever $A \in \Phi(X)$. The operator $F$ is called an upper (resp., lower) semi-Fredholm perturbation if $F + A \in \Phi_+(X)$ (resp., $F + A \in \Phi_-(X)$) whenever $A \in \Phi_+(X)$ (resp., $A \in \Phi_-(X)$).

The sets of Fredholm, upper semi-Fredholm, and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X)$, $\mathcal{F}_+(X)$, and $\mathcal{F}_-(X)$, respectively. These sets of operators were introduced and investigated in [4] (see also [12]). In particular, it is proved that $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ are closed two-sided ideals of $\mathcal{L}(X)$ while $\mathcal{F}_-(X)$ is a closed subset of $\mathcal{L}(X)$.

Our main objective here is to show that Cuthbert’s results remain valid if we replace $\mathcal{H}(X)$ by any closed two-sided ideal contained in $\mathcal{F}(X)$.

In the following, $\mathcal{J}(X)$ denotes an arbitrary nonzero closed two-sided ideal of $\mathcal{L}(X)$ satisfying

$$\mathcal{J}(X) \subseteq \mathcal{F}(X).$$

**Remark 3.2.** (1) It is worth noticing that, in general, the structure ideal of $\mathcal{L}(X)$ is extremely complicated. Most of the results on ideal structure deal with the well-known closed ideals which have arisen from applied work with operators. We can quote, for example, compact operators, weakly compact operators, strictly singular operators, strictly cosingular operators, upper semi-Fredholm perturbations, and Fredholm perturbations. In general, we have

$$\mathcal{H}(X) \subseteq \mathcal{F}(X) \subseteq \mathcal{F}_+(X) \subseteq \mathcal{F}(X),$$
$$\mathcal{H}(X) \subseteq \mathcal{C}(X) \subseteq \mathcal{F}_-(X) \subseteq \mathcal{F}(X)$$

(3.2)
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where $\mathcal{I}(X)$ and $C\mathcal{I}(X)$ denote, respectively, the ideals of $\mathcal{L}(X)$ consisting of strictly singular and strictly cosingular operators on $X$. The inclusion $\mathcal{I}(X) \subseteq \mathcal{F}_+(X)$ is due to Kato (cf. [8]) while $C\mathcal{I}(X) \subseteq \mathcal{F}_-(X)$ was proved by Vladimirskiĭ [13]..

(2) If $X$ is isomorphic to an $L_p$ space with $1 \leq p \leq \infty$ or to $C(\Xi)$ where $\Xi$ is a compact Hausdorff space, then we have

$$\mathcal{K}(X) \subseteq \mathcal{I}(X) = \mathcal{F}_+(X) = C\mathcal{I}(X) = \mathcal{F}_-(X) = \mathcal{F}(X)$$  \hspace{1cm} (3.3)

(cf. [9, equations (2.9) and (2.10)].

A Banach space $X$ is said to be an $h$-space if each closed infinite-dimensional subspace of $X$ contains a complemented subspace isomorphic to $X$ [14]. Any Banach space isomorphic to an $h$-space; $c$, $c_0$ and $l_p$ $(1 \leq p < \infty)$ are $h$-spaces. In [14, Theorem 6.2], Whitley proved that, if $X$ is an $h$-space, then $\mathcal{I}(X)$ is the greatest proper ideal of $\mathcal{L}(X)$. This, together with (3.2), implies that

$$\mathcal{K}(X) \subseteq \mathcal{F}_+(X) = \mathcal{F}(X), \quad \mathcal{K}(X) \subseteq \mathcal{F}_-(X) \subseteq \mathcal{F}(X) = \mathcal{F}(X).$$  \hspace{1cm} (3.4)

We denote by $\mathcal{O}$ the set

$$\mathcal{O} = \{ t > 0 \text{ such that } U(t) - I \in \mathcal{F}(X) \}.$$  \hspace{1cm} (3.5)

It should be noted that for a given $C_0$-semigroup, the set $\mathcal{O}$ can be empty.

**Remark 3.3.** Note that, under assumption (3.1), if $\mathcal{O} \neq \emptyset$, then the $C_0$-semigroup $(U(t))_{t \geq 0}$ can be embedded in a $C_0$-group on $X$. (It suffices to write $U(t_0) = I + [U(t_0) - I]$ for some $t_0 \in \mathcal{O}$ and to apply Theorem 2.1.) This statement improves [2, Theorem 1].

Observe that the relation

$$(U(t) - I)(U(s) - I) = (U(t + s) - I) - (U(s) - I) - (U(t) - I),$$  \hspace{1cm} (3.6)

implies that

$$s \in \mathcal{O}, \ t \in \mathcal{O} \Rightarrow s + t \in \mathcal{O}, \quad s \in \mathcal{O}, \ t \notin \mathcal{O} \Rightarrow s + t \notin \mathcal{O}. \hspace{1cm} (3.7)$$

It follows from these relations that $\mathcal{O}$ is the intersection of an additive subgroup of real number with the positive real line. Therefore, $\mathcal{O}$ may be in one of the following forms:

(i) $\mathcal{O} = ]0, \infty[;

(ii) $\mathcal{O} = \{ nx, \text{ for some } x > 0; \text{ and } n = 1, 2, \ldots \};$

(iii) $\mathcal{O}$ is a dense subset of $]0, \infty[ \text{ with empty interior}.$
The following examples taken from [2] show that all the three types of sets may occur, the above classification of $\mathcal{C}$-sets is not empty; and sets of type (ii) can arise from semigroups having bounded or unbounded infinitesimal generators.

**Examples 3.4.** Take $X = l_1$, the Banach space of absolutely convergent sequences. As mentioned above (see Remark 3.2(1)), $\mathcal{X}(X)$ is the sole closed two-sided proper ideal of $\mathcal{L}(X)$, that is, $\mathcal{X}(X) = \mathcal{F}(X)$.

1. Let $(U(t))_{t \geq 0}$ be the $C_0$-semigroup given by $U(t) = I$ for all $t \geq 0$. Clearly, for all $t > 0$, $U(t) - I \in \mathcal{X}(X)$. Accordingly, $\mathcal{C} = 0, +\infty[; and $A = 0$.

2. (a) Assume that $U(t) = \text{diag}\{e^{it}, e^{-it}, e^{it}, e^{-it}, \ldots\}$ for all $t \geq 0$. In this case, we have $\mathcal{C} = \{2n\pi, \ n = 1, 2, 3, \ldots\}$ and $A = \text{diag}\{i, -i, i, -i, \ldots\}$, the infinitesimal generator of $(U(t))_{t \geq 0}$, is bounded.

(b) Suppose now that $U(t) = \text{diag}\{e^{it}, e^{2it}, e^{3it}, e^{4it}, \ldots\}$ for all $t \geq 0$. Here, we have also $\mathcal{C} = \{2n\pi, \ n = 1, 2, 3, \ldots\}$ but $A = \text{diag}\{i, 2i, 3i, 4i, \ldots\}$, the infinitesimal generator of $(U(t))_{t \geq 0}$, is unbounded.

3. The $C_0$-semigroup $(U(t))_{t \geq 0}$ with $U(t) = \text{diag}\{e^{it}, e^{2it}, e^{3it}, \ldots, e^{nlt}, \ldots\}$ provides an example of $\mathcal{C}$-set of type (iii).

In the next theorem, we derive some relationships between the type of $\mathcal{C}$-sets and the structure of the semigroup. In particular, we show that $\mathcal{C}$ has the first form if and only if $A$ is a Fredholm perturbation. If $\mathcal{C}$ takes the third form, then $A$ is necessarily unbounded.

**Theorem 3.5.** Assume that condition (3.1) is satisfied. Then the following statements are equivalent:

(i) $\mathcal{C} = 0, +\infty[; 

(ii) $A$ is a Fredholm perturbation; 

(iii) $\lambda \mathcal{R}(\lambda, A) - I$ is a Fredholm perturbation for some (in fact for all) $\lambda > \omega$.

This result extends [2, Theorem 2] to large classes of operators which contain properly the set of compact operators.

**Proof of Theorem 3.5.** (i)$\Rightarrow$(ii). The first step in the proof of this implication consists in showing that (i) implies that $A$ is bounded. The proof of this implication is similar to that of [2, Theorem 2]. Details are omitted.

Next, since $A$ is bounded, then $U(t)$ is uniformly continuous for $t \geq 0$ (see [7]). Hence, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|U(t) - I\| < \varepsilon \text{ for } t < \delta. \tag{3.8}$$

Accordingly, for any $t < \delta$, we have

$$\left\| \frac{1}{t} \int_0^t U(s)ds - I \right\| = \left\| \frac{1}{t} \int_0^t (U(s) - I)ds \right\| \leq \frac{1}{t} \int_0^t \|U(s) - I\|ds < \varepsilon. \tag{3.9}$$
Hence, for \( \varepsilon \) small enough, \( \int_0^t U(s)\,ds \) is invertible for all \( t < \delta \). Moreover, using the identity
\[
U(t) - I = A \int_0^t U(s)\,ds,
\] (3.10)
together with the fact that \( A \) and \( U(t) \) commute, we infer that
\[
A = \left( \int_0^t U(s)\,ds \right)^{-1} (U(t) - I).
\] (3.11)

Since \( \mathcal{J}(X) \) is an ideal, we infer that \( A \in \mathcal{J}(X) \).

(ii)\(\Rightarrow\)(i). Assume that \( A \in \mathcal{J}(X) \). Using again identity (3.10) and the ideal structure of \( \mathcal{J}(X) \) we see that \( U(t) - I \in \mathcal{J}(X) \) for all \( t \geq 0 \).

(ii)\(\Rightarrow\)(iii). This follows from the identity \( \lambda R(\lambda, A) - I = AR(\lambda, A) \) and the ideal structure of \( \mathcal{J}(X) \).

(iii)\(\Rightarrow\)(ii). Assume that \( \lambda R(\lambda, A) - I \in \mathcal{J}(X) \) for all \( \lambda > \omega \). Note that the identity \( \lambda R(\lambda, A) - I = AR(\lambda, A) \) and (3.11) lead to
\[
R(\lambda, A)(U(t) - I) = (\lambda R(\lambda, A) - I) \int_0^t U(s)\,ds \quad \forall t \geq 0.
\] (3.12)
Writing (3.12) in the form
\[
\lambda R(\lambda, A)(U(t) - I) - (U(t) - I) + (U(t) - I)
= \lambda (\lambda R(\lambda, A) - I) \int_0^t U(s)\,ds \quad \forall t \geq 0,
\] (3.13)
we infer that
\[
U(t) - I = (\lambda R(\lambda, A) - I) \left[ U(t) - I + \lambda \int_0^t U(s)\,ds \right].
\] (3.14)
Next, using the fact that \( [U(t) - I + \lambda \int_0^t U(s)\,ds] \in \mathcal{L}(X) \), we get that \( U(t) - I \in \mathcal{J}(X) \) for all \( t \geq 0 \), that is, \( \emptyset = ]0, \infty[ \). This achieves the proof.

The next result asserts that if the \( \emptyset \)-set is in the form (iii), then the infinitesimal generator of \( (U(t))_{t \geq 0} \) is necessarily unbounded. It generalizes [2, Theorem 3].

**Proposition 3.6.** Assume that condition (3.1) holds true. If \( \emptyset \) is a dense subset of \( ]0, \infty[ \) with no interior points, then \( A \) is unbounded.

**Proof.** Assume, for contradiction, that \( A \) is bounded. Then, proceeding as in the proof of the implication (i)\(\Rightarrow\)(ii) in Theorem 3.5 we see that if \( t < \delta \) and \( t \in \emptyset \), then \( A \in \mathcal{J}(X) \). So, by Theorem 3.5, we get \( \emptyset = ]0, \infty[ \). This contradicts the hypothesis. \( \square \)


**Remark 3.7.** (1) Notice that if \( \mathcal{J}(X) \) is a nonzero closed two-sided ideal of \( \mathcal{L}(X) \) satisfying (3.1), then it follows from [4, Proposition 4, page 70] that

\[
\mathcal{F}_0(X) \subseteq \mathcal{J}(X) \subseteq \mathcal{F}(X), \tag{3.15}
\]

where \( \mathcal{F}_0(X) \) stands for the ideal of finite rank operators on \( X \). This shows that \( \mathcal{F}_0(X) \) is the minimal ideal (in the sense of the inclusion) in \( \mathcal{L}(X) \) for which the results of this section are valid. Evidently, if \( X \) has the approximation property, then we have \( \mathcal{F}_0(X) = \mathcal{K}(X) \).

(2) Even though the description of the ideal structure of \( \mathcal{L}(X) \) is a complex task, there exist some Banach spaces \( X \) for which \( \mathcal{L}(X) \) has only one proper nonzero closed two-sided ideal. The first result in this direction was established by Calkin (cf. [4]). He proved that if \( X \) is a separable Hilbert space, then \( \mathcal{K}(X) \) is the unique proper nonzero closed two-sided ideal of \( \mathcal{L}(X) \). An extension of this result was obtained by Gohberg et al. [4]. They proved the same result for \( X = l_p, 1 \leq p < \infty \), and \( X = c_0 \). In [6], Herman establishes the same result for a large class of Banach spaces, namely Banach spaces which have perfectly homogeneous block bases and satisfy (+) (for the definition and the meaning of the symbol (+) we refer to [6]). (Evidently, the spaces \( l_p, 1 \leq p < \infty \), and \( c_0 \) belong to this class.) Thus, if \( X \) has perfectly homogeneous block bases which satisfy (+), then

\[
\mathcal{K}(X) = \mathcal{F}_+(X) = \mathcal{F}_-(X) = \mathcal{F}(X). \tag{3.16}
\]

Consequently, for this class of spaces the results of this section use the ideal of compact operators and coincide with those obtained in [2]. Hence, for such spaces the Cuthbert results are optimal.

### 4. Further extensions.

Let \( X \) be a Banach space. An operator \( R \in \mathcal{L}(X) \) is called a Riesz operator if \( \lambda - R \in \Phi(X) \) for all scalars \( \lambda \neq 0 \). Let \( \mathcal{R}(X) \) denote the class of all Riesz operators. For further discussions concerning this family of operators, we refer to [1, 12] and the references therein. For our purpose, we recall that Riesz operators satisfy the Riesz-Schauder theory of compact operators, \( \mathcal{R}(X) \) is not an ideal of \( \mathcal{L}(X) \) [1], and \( \mathcal{F}(X) \) is the largest ideal contained in \( \mathcal{R}(X) \) [12]. Hence the sets \( \mathcal{K}(X), \mathcal{J}(X), \mathcal{C}(X), \mathcal{F}(X) \), and \( \mathcal{F}_+(X) \) are also contained in \( \mathcal{R}(X) \).

Let \( A \in \mathcal{L}(X) \). The Fredholm region of \( A \) is defined as \( \{ \lambda \in \mathbb{C}; \lambda - A \in \Phi(X) \} \) and denoted by \( \Phi_A \). Next, let \( \Phi^0_A := \{ \lambda \in \Phi_A : i(\lambda - A) = 0 \} \) and define the set

\[
\sigma_b(A) := \mathbb{C} \setminus \rho_b(A), \tag{4.1}
\]

where

\[
\rho_b(A) := \{ \lambda \in \Phi^0_A \text{ such that all scalars near } \lambda \text{ are in } \rho(A) \}. \tag{4.2}
\]

Following [5, 11], \( \sigma_b(\cdot) \) is called the Browder essential spectrum.
We say that an operator $F \in \mathcal{L}(X)$ is polynomially compact (see [3]) if there is a nonzero complex polynomial $p(z)$ such that the operator $p(F)$ is compact. We designate by $\mathcal{P}(X)$ the set of polynomially compact operators on $X$. Let $F \in \mathcal{P}(X)$, the nonzero polynomial $p(z)$ of least degree and leading coefficient 1 such that $p(F)$ is compact will be called the minimal polynomial of $F$. We denote by $\mathfrak{P}(X)$ the subset of $\mathcal{P}(X)$ defined by

$$\mathfrak{P}(X) := \left\{ F \in \mathcal{P}(X) \text{ such that the minimal polynomial of } F ight\}.$$  

(4.3)

We first prove the following lemma which is required in the sequel.

**Lemma 4.1.** If $F \in \Xi(X)$, then $I + F \in \Phi(X)$ and $i(I + F) = 0$.

**Proof.** Since $p(F) \in \Xi(X)$ ($p(\cdot)$ denotes the minimal polynomial of $F$), then $\sigma_b(p(F)) = \{0\}$. By hypothesis $p(-1) \neq 0$, then $p(-1) \notin \sigma_b(p(F))$. Next, making use of the spectral mapping theorem for the Browder essential spectrum [5, Theorem 4] we conclude that $-1 \notin \sigma_b(F)$, that is, $-1 \in \rho_b(F)$. This ends the proof. $\square$

The developments below are mainly suggested by the fact that, in general, the sets $\mathcal{P}(X)$ and $\Xi(X)$ do not coincide. Indeed, if $p(z) = (z - \lambda_1)^{n_1} \cdots (z - \lambda_k)^{n_k}$ is the minimal polynomial of $F \in \Xi(X)$, then, by the structure theorem of Gilfeather [3, Theorem 1], the spectrum of $F$ consists of countably many points with $\{\lambda_1, \ldots, \lambda_k\}$ as only possible limit points and such that all but possibly $\{\lambda_1, \ldots, \lambda_k\}$ are eigenvalues with finite-dimensional generalized eigenspaces. This, together with the fact that the operators belonging to $\mathfrak{P}(X)$ satisfy the Riesz-Schauder theory of compact operators (see above), implies that $\mathfrak{P}(X) \neq \Xi(X)$. Thus the next result improves Proposition 3.6.

**Proposition 4.2.** Let $(U(t))_{t \geq 0}$ be a $C^0$-semigroup on $X$. If

$$\{ t > 0 \text{ such that } U(t) - I \in \Xi(X) \} \neq \emptyset,$$

(4.4)

then $(U(t))_{t \geq 0}$ can be embedded in a $C_0$-group on $X$.

**Proof.** By hypothesis, there exists $t_0$ such that $U(t_0) - I \in \Xi(X)$. Since $U(t_0) = I + [U(t_0) - I]$, the use of Lemma 4.1 implies that $U(t_0) \in \Phi(X)$. Now, the result follows from Theorem 2.1. $\square$

Due to some technical difficulties, we do not know whether or not Theorem 3.5 is valid for perturbations belonging to $\Xi(X)$. So, we discuss this result for a subset of $\Xi(X)$ consisting of power compact operators, that is,

$$\mathcal{P}(X) := \{ F \in \mathcal{L}(X) \text{ such that } F^n \in \Xi(X) \text{ for some integer } n \geq 1 \}.$$  

(4.5)
Our principal motivation here rely on the fact that, for some classes of Banach spaces, we have $\mathcal{F}(X) \subseteq \mathcal{P}(X)$. In particular, if $X$ is isomorphic to an $L_p$ space with $1 \leq p \leq \infty$ or to $C(\Omega)$ where $\Omega$ is a metric compact Hausdorff space, then $\mathcal{F}(X) = \mathcal{F}(X)$ (cf. (3.3)). Moreover, by [10, Theorem 1], we have $\mathcal{F}(X) \subseteq \mathcal{H}(X)$. These conclusions are also valid if $X$ is isomorphic to an $L_p$ space with $1 \leq p < \infty$ and $c_0$ [6]. Note also that if $X$ has the Dunford-Pettis property (a Banach space $X$ is said to have the Dunford-Pettis property if for every Banach space $Y$ every weakly compact operator $T : X \to Y$ takes weakly compact sets in $X$ into relatively norm compact sets of $Y$), then $\mathcal{F}(X) = \mathcal{H}(X)$ (cf. (3.3)). Moreover, by [10, Theorem 1], we have $\mathcal{F}(X) \subseteq \mathcal{H}(X)$. These conclusions are also valid if $X$ is an $L_p$ space with $1 \leq p < \infty$ and $c_0$ [6].

In the light of these observations, we project to extend Theorem 3.5 to semi-groups $\{U(t)\}_{t \geq 0}$ for which there exists $t_0 > 0$ such that $U(t_0) - I \in \mathcal{F}(X)$. Evidently, since $\mathcal{F}(X) \subseteq \mathcal{H}(X)$, Proposition 4.2 holds also true for power compact perturbations. More precisely, we have the following theorem.

**Theorem 4.3.** Let $\{U(t)\}_{t \geq 0}$ be a $C_0$-semigroup on $X$ with type $\omega$ and let $A$ denote its infinitesimal generator. Define the set $\mathcal{C}$ by

$$\mathcal{C} = \{ t \geq 0 \text{ such that } U(t) - I \in \mathcal{P}(X) \}.$$  

Then, the following items are equivalent:

(i) $\mathcal{C} = [0, +\infty[$;

(ii) $A \in \mathcal{P}(X)$;

(iii) $[\lambda R(\lambda, A) - I] \in \mathcal{P}(X)$ for some (in fact for all) $\lambda > \omega$.

**Proof.** We try to imitate the procedure in the proof of Theorem 3.5. Let us first observe that if $U(t) - I \in \mathcal{P}(X)$, then there exists $m \geq 1$ such that $(U(t) - I)^m \in \mathcal{H}(X)$. Using the spectral mapping theorem (see, e.g., [15, page 227]), one sees that that spectrum of $U(t) - I$ is either finite or a countable set accumulating only at zero. Moreover,

$$\sigma(U(t) - I) = \sigma(U(t)) - 1.$$  

This means that, apart possibly from the point 1, $\sigma(U(t)) = \{ e^{\eta t} : \eta \in P\sigma(A) \}$ ($P\sigma(A)$ stands for the point spectrum of $A$) and, for any $\varepsilon > 0$, the set $\{ \lambda \in \sigma(U(t)) : |\lambda - 1| > \varepsilon \}$ is finite for all $t > 0$. Then arguing as in the proof of [2, Theorem 2], we conclude that (i) implies that $A \in \mathcal{L}(X)$. Furthermore, similar arguments as in the proof of Theorem 3.5 [(i) $\Rightarrow$ (ii)] imply that

$$A = \left[ \int_0^t U(s)ds \right]^{-1} (U(t) - I) = (U(t) - I) \left[ \int_0^t U(s)ds \right]^{-1}$$

which leads to $A \in \mathcal{P}(X)$.
The remainder of the proof is verbatim that of Theorem 3.5. It suffices to use the fact that $U(t) - I$ and $[\int_0^t U(s)ds]^{-1}$ (resp., $A$ and $R(\lambda, A)$) commute.

We close this section by noticing that Proposition 3.6 is also valid for power compact perturbations. The proof uses Theorem 4.3.

**References**


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