BIHARMONIC CURVES IN MINKOWSKI 3-SPACE

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We give a differential geometric interpretation for the classification of biharmonic curves in semi-Euclidean 3-space due to Chen and Ishikawa (1991).

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1. Introduction. Chen and Ishikawa [1] classified biharmonic curves in semi-Euclidean space \( E^n_\nu \). They showed that every biharmonic curve lies in a 3-dimensional totally geodesic subspace. Thus, it suffices to classify biharmonic curves in semi-Euclidean 3-space.

In this note, we point out that every biharmonic Frenet curve in Minkowski 3-space \( E^3_1 \) is a helix whose curvature \( \kappa \) and torsion \( \tau \) satisfy \( \kappa^2 = \tau^2 \).

2. Preliminaries. Let \((M^3,h)\) be a time-oriented Lorentz 3-manifold. Let \( \gamma: I \to M \) be a unit speed curve. Namely, the velocity vector field \( \gamma' \) satisfies \( h(\gamma',\gamma') = \varepsilon_1 = \pm 1 \). The constant \( \varepsilon_1 \) is called the causal character of \( \gamma \). A unit speed curve is said to be spacelike or timelike if its causal character is 1 or \(-1\), respectively.

A unit speed curve \( \gamma \) is said to be a geodesic if \( \nabla \gamma' \gamma' = 0 \). Here, \( \nabla \) is the Levi-Civita connection of \((M,h)\).

A unit speed curve \( \gamma \) is said to be a Frenet curve if \( h(\gamma'',\gamma''') \neq 0 \). Like Euclidean geometry, every Frenet curve \( \gamma \) in \((M,h)\) admits a Frenet frame field along \( \gamma \). Here, a Frenet frame field \( P = (p_1,p_2,p_3) \) is an orthonormal frame field along \( \gamma \) such that \( p_1 = \gamma'(s) \) and \( P \) satisfies the following Frenet-Serret formula (cf. [2]; see also [4, 5]):

\[
\nabla_{\gamma'} P = P \begin{pmatrix}
0 & -\varepsilon_1 \kappa & 0 \\
\varepsilon_2 \kappa & 0 & \varepsilon_2 \tau \\
0 & -\varepsilon_3 \tau & 0
\end{pmatrix}.
\]

(2.1)

The functions \( \kappa \geq 0 \) and \( \tau \) are called the curvature and torsion, respectively. The vector fields \( p_1, p_2, \) and \( p_3 \) are called tangent vector field, principal normal vector field, and binormal vector field of \( \gamma \), respectively. The constants \( \varepsilon_2 \) and \( \varepsilon_3 \) defined by

\[
\varepsilon_i = h(p_i,p_i), \quad i = 2, 3
\]

(2.2)
are called second causal character and third causal character of $\gamma$, respectively. Note that $\varepsilon_3 = -\varepsilon_1 \cdot \varepsilon_2$.

As in the case of Riemannian geometry, a Frenet curve $\gamma$ is a geodesic if and only if $\kappa = 0$.

A Frenet curve with constant curvature and zero torsion is called a pseudo-circle.

A helix is a Frenet curve whose curvature and torsion are constants. Pseu-docircles are regarded as degenerate helices. Helices, which are not circles, are frequently called proper helices.

The mean curvature vector field $H$ of a unit speed curve $\gamma$ is $H = \varepsilon_1 \nabla_{\gamma'} \gamma'$. If $\gamma$ is a Frenet curve, then $H$ is given by

$$H = -\varepsilon_3 \kappa p_2.$$ (2.3)

To close this section, we recall the notion of biharmonicity for unit speed curves.

Let $\gamma = \gamma(s)$ be a unit speed curve in a Lorentz 3-manifold $(M, h)$ defined on an interval $I$. Denote by $\gamma^* TM$ the vector bundle over $I$ obtained by pulling back the tangent bundle $TM$:

$$\gamma^* TM := \bigcup_{s \in I} T_{\gamma(s)} M.$$ (2.4)

The Laplace operator $\Delta$ acting on the space $\Gamma(\gamma^* TM)$ of all smooth sections of $\gamma^* TM$ is given explicitly by

$$\Delta = -\varepsilon_1 \nabla_{\gamma'} \nabla_{\gamma'}.$$ (2.5)

**Definition 2.1.** A unit speed curve $\gamma : I \to M$ in a Lorentz 3-manifold $M$ is said to be biharmonic if $\Delta H = 0$.

If $M$ is the semi-Euclidean 3-space, then $\gamma$ is biharmonic if and only if $\Delta \Delta \gamma = 0$.

3. Biharmonic curves. Chen and Ishikawa classified biharmonic curves in semi-Euclidean 3-space. In particular, they showed that in Euclidean 3-space, there are no proper biharmonic curves (i.e., biharmonic curves which are not harmonic). On the other hand, in indefinite semi-Euclidean 3-space, there exist proper biharmonic curves. Here, we recall their classification theorem.

**Theorem 3.1** (see [1]). Let $\gamma$ be a spacelike curve in indefinite semi-Euclidean 3-space $E^3_1$. Then, $\gamma$ is biharmonic if and only if $\gamma$ is congruent to one of the following:

1. a spacelike line;
2. a spacelike curve $\gamma(s) = (as^3 + bs^2, as^3 + bs^2, s)$ in $E^3_1$, where $a$ and $b$ are constants such that $a^2 + b^2 \neq 0$;
(3) a spacelike curve \( \gamma(s) = (a^2 s^3/6, as^2/2, -a^2 s^3/6 + s) \) in \( E_3^1 \), where \( a \) is a nonzero constant;
(4) a spacelike curve \( \gamma(s) = (a^2 s^3/6, as^2/2, a^2 s^3/6 + s) \) in \( E_3^2 \), where \( a \) is a nonzero constant.

To give a differential geometric interpretation of the above result, we need to start with the following general result (cf. [2]).

**Theorem 3.2.** Let \( \gamma : I \to M \) be a Frenet curve in a Lorentz 3-manifold \((M,h)\). Denote by \( \Delta \) the Laplace operator acting on \( \Gamma(\gamma^*TM) \). Then, \( \gamma \) satisfies \( \Delta H = \lambda H \) if and only if \( \gamma \) is a helix (including a geodesic). In this case, the eigenvalue \( \lambda \) is \( \lambda = -\varepsilon_3 (\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2) \).

**Proof.** Direct computation shows that
\[
\Delta H = -3\varepsilon_3 \kappa \kappa' \mathbf{p}_1 - \varepsilon_2 \{ \kappa'' - \varepsilon_2 \kappa (\varepsilon_1 \kappa^2 - \varepsilon_3 \tau^2) \} \mathbf{p}_2 - \varepsilon_1 (2\kappa' \tau + \kappa \tau') \mathbf{p}_3. 
\] (3.1)
Thus, \( \Delta H = \lambda H \) if and only if
\[
\kappa \kappa' = 0, \quad 2\kappa' \tau + \kappa \tau = 0, \quad \kappa'' - \varepsilon_2 \kappa (\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2) = -\varepsilon_1 \lambda \kappa. 
\] (3.2)
These formulae imply that \( \gamma \) is a spacelike or timelike helix whose curvature and torsion satisfy \( \lambda = -\varepsilon_3 (\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2) \).

**Theorem 3.2** implies the following two results.

**Corollary 3.3.** Let \( \gamma \) be a Frenet curve in a Lorentz 3-manifold \((M,h)\). Then, \( \gamma \) is a nongeodesic biharmonic curve if and only if it is one of the following:
(1) \( \gamma \) is a spacelike helix with a spacelike principal normal such that \( \kappa = \pm \tau \);
(2) \( \gamma \) is a timelike helix such that \( \kappa = \pm \tau \).

Note that there exist no biharmonic spacelike curves in \( M \) with spacelike principal normals.

**Corollary 3.4.** Let \( \gamma \) be a Frenet curve in \((M,h)\). Then, \( \gamma \) is a helix if and only if
\[
\nabla_{\gamma'} \nabla_{\gamma'} \kappa H \gamma' = 0 \quad (3.3)
\] for some constant \( \mathcal{H} \). In this case, the constant \( \mathcal{H} \) equals \( -\varepsilon_2 (\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2) \).

Note that Ikawa obtained **Corollary 3.4** for timelike curves (see [3, Proposition 4.1]). Thus, we give here an analytic meaning of (3.3). Since we treat both spacelike and timelike curves in **Corollary 3.4**, we get a generalisation of [3, Proposition 4.1].

In the case where \( M \) is the Minkowski 3-space \( E_3^1 \), it is known that helices with \( \tau = \pm \kappa \neq 0 \) are cubic curves, and one can explicitly give the formula of such helices (see, e.g., Kobayashi [6]). Moreover, it is easy to check that such spacelike helices are congruent to the curves given in **Theorem 3.1**.
Now, we rephrase the classification due to Chen and Ishikawa. Since case (4) in Theorem 3.1 is the image of a timelike helix satisfying $\kappa^2 = \tau^2 = a^2$ under the following anti-isometry from $\mathbb{E}_1^3$ onto $\mathbb{E}_2^3$:

$$\mathbb{E}_1^3 \ni (u,v,w) \mapsto (w,v,u),$$

we may restrict our attention to curves in Minkowski 3-space $\mathbb{E}_1^3$.

**Proposition 3.5.** Let $\gamma$ be a unit speed curve in Minkowski 3-space $\mathbb{E}_1^3$. Then, $\gamma$ is biharmonic if and only if $\gamma$ is congruent to one of the following:

1. a spacelike or timelike line;
2. a spacelike curve such that $h(\gamma'', \gamma'') = 0$ is given by

$$\gamma(s) = \left(as^3 + bs^2, as^3 + bs^2, s\right),$$

where $a$ and $b$ are constants such that $a^2 + b^2 \neq 0$;
3. a spacelike helix with a spacelike principal normal vector field satisfying $\kappa^2 = \tau^2 = a^2$;

$$\gamma(s) = \left(\frac{a^2 s^3}{6}, \frac{as^2}{2}, \frac{-a^2 s^3}{6} + s\right);$$

4. a timelike helix satisfying $\kappa^2 = \tau^2 = a^2$;

$$\gamma(s) = \left(\frac{a^2 s^3}{6} + s, \frac{as^2}{2}, \frac{a^2 s^3}{6}\right).$$

**References**


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