MODULUS OF SMOOTHNESS AND THEOREMS CONCERNING APPROXIMATION ON COMPACT GROUPS

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We consider the generalized shift operator defined by 

\[(Sh_u f)(g) = \int_G f(tut^{-1}g) dt\]

on a compact group \(G\), and by using this operator, we define "spherical" modulus of smoothness. So, we prove Stechkin and Jackson-type theorems.

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1. Introduction. In this paper, we prove some theorems on absolutely convergent Fourier series in the metric space \(L^2(G)\), where \(G\) is a compact group. The algebra of absolutely convergent Fourier series is a subject matter about which a good deal, although far from everything, is known (see [5, page 328]). Like many branches of harmonic analysis on \(T\) and \(R\), the theory of absolutely convergent Fourier series is a fruitful source of questions about the corresponding entity for compact groups. By using some absolute convergence theorems of the classical Fourier series, (see [1, 11]), a generalized form of Stechkin [6] and Szasz theorem [1, 11] of the Fourier series on compact groups is obtained. Thus, we solve open problems formulated in [5, page 366] (see also [3, Chapter I, page 9]).

2. Preliminaries and notation. Now, we explain some of the notation and terminologies used throughout the paper.

Let \(G\) be a compact group with a dual space \(\hat{G}\), \(dg\) denote the Haar measure on \(G\) normalized by the condition \(\int_G dg = 1\), and \(\int_G f(g) dg\) denote the Haar integral of a function \(f\) on \(G\). Let \(U_\alpha, \alpha \in \hat{G}\) denotes the irreducible unitary representation of \(G\) in the finite dimensional Hilbert space \(V_\alpha\). We reserve the symbol \(d_\alpha\) for the dimension of \(U_\alpha\). Thus, \(d_\alpha\) is a positive integer. Also, we denote by \(\chi_\alpha\) and \(t_{ij}^\alpha (i,j = 1,2,\ldots,d_\alpha), \alpha \in \hat{G}\) the character and matrix elements (coordinate functions) of \(U_\alpha\), respectively.

Let \(L^p(G)\) be the space of all functions \(f\) equipped with the norm

\[\|f\|_p = \left\{\int_G |f(g)|^p dg\right\}^{1/p}.
\]

We write \(\|\cdot\|_p\) instead of \(\|\cdot\|_{L^p(G)}\), and \(L^\infty = C\) is the corresponding space of continuous functions, and \(\|f\| = \max\{|f(g)| : g \in G\}\). As it is known (see [4]
or [10, page 99], the space $L_2(G)$ can be decomposed into the sum

$$L_2(G) = \sum_{\alpha \in \hat{G}} \Phi H_\alpha,$$

where

$$H_\alpha = \{ f \in C(G) : f(g) = \text{tr}(U_\alpha(g)C), \ C = \text{Hom}(V_\alpha, V_\alpha) \}.$$  (2.3)

This theorem is one of the most important results of the harmonic analysis on compact groups. The orthogonal projection $Y_\alpha : L_2(G) \to H_\alpha$ is given by

$$(Y_\alpha f)(g) = d_\alpha \int_G f(h) \chi_\alpha(gh^{-1}) dh,$$

where $(Y_\alpha f)(g)$ does not depend on the choice of a basis in $L_2$. Carrying out this construction for every space $H_\alpha$, $\alpha \in \hat{G}$, we obtain an orthonormal basis in $L_2$ consisting of the functions $\sqrt{d_\alpha} t_{ij}^{\alpha}(g)$, $\alpha \in \hat{G}$, $1 \leq i, j \leq d_\alpha$. Any function $f \in L_2(G)$ can be expanded into a Fourier series with respect to this basis

$$f(g) = \sum_{\alpha \in \hat{G}} \sum_{i,j=1}^{d_\alpha} a_{ij}^{\alpha} t_{ij}^{\alpha}(g),$$

where the Fourier coefficients $a_{ij}^{\alpha}$ are defined by the following relations:

$$a_{ij}^{\alpha} = d_\alpha \int_G f(g) t_{ij}^{\alpha}(g) dg,$$

such that $\overline{t_{ij}^{\alpha}(g)} = t_{ij}^{\alpha}(g^{-1})$, where $g^{-1}$ is the inverse of $g$. Note that (2.5) is a convergent series in the mean and that the Parseval’s equality

$$\int_G |f(g)|^2 dg = \sum_{\alpha \in \hat{G}} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} |a_{ij}^{\alpha}|^2$$

holds. The aforementioned result of harmonic analysis on a compact group can be found, for example, in [4, 5, 7, 10].

We denote by $S_h$ the generalized translation operator on compact group $G$ defined by

$$(S_h f)(g) = \int_G f(tu^{-1}g) dt,$$

$$\Delta_h f(g) = f(g) - (S_h f)(g) = (E - S_h) f,$$  (2.8)
where \( u, g \in G \) and \( E \) is the identity operator. We set

\[
\Delta^k u f = \Delta u \left( \Delta^{k-1} u f \right) = (E - Sh_u)^k f = \sum_{i=0}^{k} (-1)^{k+i} C_k^i Sh^i_u f,
\]

(2.9)

in which \( Sh^0_u f = f \) and \( Sh_u (Sh^i_u f) = Sh^i_u f, \, i = 1, 2, \ldots, k \) and \( k \in \mathbb{N} \).

We note that \( \alpha \) is a complicated index. Since \( \hat{G} \) is a countable set, there are only countably many \( \alpha \in \hat{G} \) for which \( \alpha_{ij} \neq 0 \) for some \( i \) and \( j \); enumerate them as \( \{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\} \). So, \( d_{\alpha_0} < d_{\alpha_1} < \cdots < d_{\alpha_n} < \cdots \). Because of that, the symbol “\( \alpha < n \)” is interpreted as \( \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \subset \hat{G} \), and \( \alpha \geq n \) denotes the set \( \hat{G} \setminus (\alpha < n) \). Let \( d_{\alpha} \) as usual, be the dimension of \( U_{\alpha} \). For typographical convenience, we write \( d_n \) for the dimension of the representation \( U^{\alpha_n}, \, n = 1, 2, \ldots \). (See [5, page 458].)

We denote by \( E_n(f)_p \) the approximation of the function \( f \in L_p(G) \) by “Spherical” polynomials of degree not greater than \( n \):

\[
E_n(f)_p = \inf \left\{ \| f - T_n \|_p : T_n \in \sum_{\alpha < n, \alpha \in \hat{G}} \oplus H_\alpha \right\}.
\]

(2.10)

The sequence of best approximations \( \{E_n(f)_p\}_{n=0}^\infty \) is a constructive characteristic of the function \( f \). In the capacity of structural characteristic of the function \( f \) on a compact group \( G \), we define its Spherical modulus of smoothness of order \( k \) by

\[
\omega_k(f; \tau)_p = \sup \left\{ \| (E - Sh_u)^k f \|_p : u \in W_\tau \right\},
\]

(2.11)

where \( W_\tau \) is a neighborhood of \( e \) in \( G \). In other words,

\[
W_\tau = \{u : \rho(u, e) < \tau, \, u \in G\},
\]

(2.12)

where \( \rho \) is a pseudometric on \( G \) and \( \tau \) is any positive real number. It is easy to show the following properties of \( \omega_k(f, \tau)_p \):

(a) \( \lim_{\tau \to 0} \omega_k(f, \tau)_p = 0 \);

(b) \( \omega_k(f, \tau)_p \) is a continuous monotonically increasing function with respect to \( \tau \);

(c) \( \omega_k(f_1 + f_2, \tau)_p \leq \omega_k(f_1, \tau)_p + \omega_k(f_2, \tau)_p \);

(d) \( \omega_{k+l}(f, \tau)_p \leq 2^l \omega_k(f, \tau)_p, \, l = 1, 2, \ldots \).

3. Main results. We need the following simple but useful lemma.
Lemma 3.1. The following equality holds for all \( u, g \in G \):

\[
\left( \text{Sh}_u t_{ij}^\alpha \right) (g) = \frac{X_\alpha(u)}{d_\alpha} t_{ij}^\alpha (g). \quad (3.1)
\]

Proof. Using the orthogonality relations and other formulas for matrix elements \( t_{ij}^\alpha(g) \) (see [7, page 189]), we have

\[
\int_G t_{ij}^\alpha(tut^{-1}g)dt = \frac{d_\alpha}{d_\alpha} \sum_{p=1}^{d_\alpha} \sum_{q=1}^{d_\alpha} t_{qp}(u)t_{ij}^\alpha(g) \int_G t_{iq}(t)t_{qj}(t)dt
\]

\[
= \frac{1}{d_\alpha} \sum_{p=1}^{d_\alpha} t_{pp}(u)t_{ij}^\alpha(g) = \frac{1}{d_\alpha} X_\alpha(u)t_{ij}^\alpha(g). \quad (3.2)
\]

This proves the lemma. \( \square \)

The following formula is the particular event of the above lemma:

\[
\int_G X_\alpha(tut^{-1}g)dt = \frac{X_\alpha(u)X_\alpha(g)}{d_\alpha}. \quad (3.3)
\]

It can be called a Weyl formula.

We note that the expansion (2.5) is connected with the expansion

\[
f(g) = \sum_{\alpha \in \hat{G}} Y_\alpha(f)(g), \quad Y_\alpha \in H_\alpha, \quad (3.4)
\]

which is defined by (2.4), that is, by the equality

\[
Y_\alpha(f)(g) = \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha t_{ij}^\alpha(g). \quad (3.5)
\]

Thus, the coefficients \( a_{ij}^\alpha \) are defined by (2.6). Using Lemma 3.1 and the definition of \( Y_\alpha \), we obtain

\[
Y_\alpha(\text{Sh}_u f)(g) = \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha \int_G t_{ij}^\alpha(tut^{-1}g)dt
\]

\[
= \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha \frac{X_\alpha(u)}{d_\alpha} t_{ij}^\alpha(g)
\]

\[
= \frac{X_\alpha(u)}{d_\alpha} Y_\alpha(f)(g). \quad (3.6)
\]

The following are simple facts with frequent usage: if \( f \in L_p \), then

1. \( \| \text{Sh}_u f \|_p \leq \| f \|_p \);
2. \( \| f - \text{Sh}_u f \|_p \to 0 \) as \( u \to e \);
3. \( (Y_\alpha(\text{Sh}_u f)) (g) = (X_\alpha(u)/X_\alpha(e))(Y_\alpha f)(g) \) for all \( \alpha \in \hat{G} \).

We note that \( X_\alpha(e) = d_\alpha \).
Theorem 3.2. If \( f \in L_2 \) and \( f \) is not constant, then
\[
E_n(f)_2 \leq \sqrt{\frac{d_n}{d_n-2k}} \omega_k \left( f; \frac{1}{n} \right)_2, \quad n = 1, 2, \ldots
\]  
(3.7)

Proof. Let \( f \in L_2 \) and \( S_n(f, g) \) denote the \( n \)th partial sum of the Fourier series (2.5), that is,
\[
S_n(f, g) = \sum_{\alpha<n} \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha f_{ij}(g) = \sum_{p=0}^{n} \sum_{i,j=1}^{d_{ap}} a_{ij}^{ap} f_{ij}(g), \quad n = 1, 2, \ldots
\]  
(3.8)

Using Parseval’s equality for the compact group \( G \), we have
\[
E_n(f)_2 = \|f - S_n(f)\|^2_2 = \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left| a_{ij}^\alpha \right|^2.
\]  
(3.9)

Using (3), it is not hard to see that
\[
(Y_\alpha(\Delta^k f))(g) = \left(1 - \frac{X_\alpha(u)}{d_\alpha}\right)^k (Y_\alpha f)(g), \quad \alpha \in \hat{G}.
\]  
(3.10)

Consequently, \( (\Delta^k f)(g) = \sum_{\alpha \in \hat{G}} (1 - \chi_\alpha(u)/d_\alpha)^k a_{ij}^\alpha \). By another application of Parseval’s equality, we obtain
\[
\|\Delta^k u f\|^2_2 \geq \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left| 1 - \frac{X_\alpha(u)}{d_\alpha} \right|^2 \left| a_{ij}^\alpha \right|^2 \geq \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left| 1 - \frac{X_\alpha(u)}{d_\alpha} \right|^{2k} \left| a_{ij}^\alpha \right|^2
\]
\[
= \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left( 1 - \frac{2k \text{Re} X_\alpha(u)}{d_\alpha} + \frac{|X_\alpha(u)|^2}{d_\alpha^2} \right)^k \left| a_{ij}^\alpha \right|^2.
\]  
(3.11)

Now, using Bernolly’s inequality \( (1+x)^k \geq 1 + kx \) for \( x \geq -1 \), we obtain
\[
\|\Delta^k u f\|^2_2 \geq \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left( 1 - \frac{2k \text{Re} X_\alpha(u)}{d_\alpha} + \frac{k|X_\alpha(u)|^2}{d_\alpha^2} \right) \left| a_{ij}^\alpha \right|^2.
\]  
(3.12)

Consequently,
\[
\|\Delta^k u f\|^2_2 \geq \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left| a_{ij}^\alpha \right|^2 - \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \frac{2k \text{Re} X_\alpha(u)}{d_\alpha} \left| a_{ij}^\alpha \right|^2;
\]  
(3.13)

therefore,
\[
E_n(f)_2 \leq \|\Delta^k u f\|^2_2 + 2k \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \frac{\text{Re} X_\alpha(u)}{d_\alpha} \left| a_{ij}^\alpha \right|^2.
\]  
(3.14)
Let $\Phi_{W_T}$ be a nonnegative integrable function vanishing outside $W_T$ and satisfying the condition $\int_G \Phi_{W_T}(g) dg = 1$. For example, we can take $\Phi_{W_T} = \xi_{W_T}/\mu(W_T)$, where $\mu(W_T)$ is the Haar measure of $W_T$ and $\xi_{W_T}$ is the characteristic function of $W_T$. Multiplying both sides of (3.14) by $\Phi_{W_1/n}$, and integrating with respect to $u$ on $G$, and using the equality $\int_G |\chi_{\alpha}(u)|^2 dg = 1$ (see [7, page 195]), we obtain

$$\int_G E^2_n(f) 2 \Phi_{W_1/n}(u) du \leq \int_G \left\| \Delta^k_{u} f \right\|_2^2 \Phi_{W_1/n} du$$

$$+ 2k \sum_{\alpha \in \hat{G}} \frac{1}{d_{\alpha}} \sum_{i,j=1}^{d_{\alpha}} \left| a_{\alpha}^{ij} \right|^2 \int_G |\chi_{\alpha}(u)| \Phi_{W_1/n}(u) du$$

$$\leq \sup \left\| \Delta^k_{u} f \right\|_2^2 + \frac{2k}{d_n} \sum_{\alpha \in \hat{G}} \frac{1}{d_{\alpha}} \sum_{i,j=1}^{d_{\alpha}} \left| a_{\alpha}^{ij} \right|^2.$$  (3.15)

Therefore, it is not hard to see that

$$E^2_n(f) 2 \leq \omega_k^2 \left( f, \frac{1}{n} \right) + \frac{2k}{d_n} E^2_n(f) 2.$$  (3.16)

Finally, we obtain

$$E_n(f) 2 \leq \sqrt{\frac{d_n}{d_n - 2k}} \omega_k \left( f, \frac{1}{n} \right),$$  (3.17)

which proves the theorem.

This theorem is given without proof in [8] for the case where $k = 1$.

We note that the matrix elements of unitary representations $t_{ij}^{\alpha}(g)$ satisfy the relations

$$\sum_{j=1}^{d_{\alpha}} t_{ij}^{\alpha}(g) t_{kj}^{\alpha}(g) = \sum_{j=1}^{d_{\alpha}} t_{ij}^{\alpha}(g) t_{jk}^{\alpha}(g) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$  (3.18)

In particular, we have

$$\sum_{j=1}^{d_{\alpha}} \left| t_{ij}^{\alpha}(g) \right|^2 = 1 \Rightarrow \left| t_{ij}^{\alpha}(g) \right| \leq 1$$  (3.19)

for all $\alpha \in \hat{G}$ and $i, j = 1, 2, \ldots, d_{\alpha}$. Furthermore, it is obvious that $|a_{ij}^{\alpha} t_{ij}^{\alpha}(g)| \leq |a_{ij}^{\alpha}|$; therefore, according to the sufficient condition for absolutely convergent Fourier series on the group $G$, the series $\sum_{\alpha \in \hat{G}} \sum_{i,j=1}^{d_{\alpha}} |a_{ij}^{\alpha}|$ is convergent. Let $A(G) := \{ f : \sum_{\alpha \in \hat{G}} \sum_{i,j=1}^{d_{\alpha}} |a_{ij}^{\alpha}| < +\infty \}$. Using Theorem 3.2, and repeating the proof of analogous theorems (see [1, Chapter IX] or [6, Chapter II]) with some changes, we obtain the following theorems.
**Theorem 3.3.** If \( f(g) \in L_2(G) \), then
\[
\sum_{n=1}^{\infty} \frac{\omega_k(f, 1/n)_2}{\sqrt{n}} < +\infty \implies f(g) \in A(G).
\] (3.20)

This theorem is analogous to the Szasz theorem of the classical Fourier series in the case where \( k = 1 \) and \( G = T \).

**Theorem 3.4.** If \( f(g) \in L_2(G) \), then
\[
\sum_{n=1}^{\infty} \frac{E_n(f)_2}{\sqrt{n}} < +\infty \implies f(g) \in A(G).
\] (3.21)

This theorem is also analogous to a theorem in trigonometric case proved by Stechkin [9].

4. Applications to compact group \( SU(2) \). The group \( SU(2) \) consists of unimodular unitary matrices of the second order, that is, matrices of the form
\[
u = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1. \tag{4.1}
\]
Therefore, each element \( u \) of \( SU(2) \) is uniquely determined by a pair of complex numbers \( \alpha \) and \( \beta \) such that \( |\alpha|^2 + |\beta|^2 = 1 \). We have (see [5]) the relation 
\[(\alpha, \beta) \rightarrow (\phi, \theta, \psi),\]
where \( \alpha \beta \neq 0 \), \( |\alpha|^2 + |\beta|^2 = 1 \), and the parameters \( \phi \), \( \theta \), and \( \psi \) are called Euler angles defined by
\[
|\alpha| = \cos \frac{\theta}{2}; \quad \text{Arg} \alpha = \frac{\phi + \psi}{2}; \quad \text{Arg} \beta = \frac{\phi - \psi}{2}. \tag{4.2}
\]
Let \( \phi \), \( \theta \), and \( \psi \) satisfy the conditions
\[0 \leq \phi < 2\pi, \quad 0 \leq \theta < \pi, \quad -2\pi \leq \psi < 2\pi. \tag{4.3}\]
Also, we know that the dimension of the representation \( T^l \) of \( SU(2) \) is equal to \( 2l + 1 \), where \( l = 0, 1/2, 1, \ldots \) and the matrix elements of \( T^l \) for group \( SU(2) \) are defined by
\[
t^l_{mn}(u) = e^{-(n\psi + m\phi)}p^l_{mn}(\cos \theta)i^{(m-n)}. \tag{4.4}
\]
Expressing \( t^l_{mn}(u) \) in terms of \( p^l_{mn}(\cos \theta) \), we arrive at the following conclusion:
Any function \( f(\phi, \theta, \psi) \), \( 0 \leq \phi < 2\pi, 0 \leq \theta < \pi \), and \( -2\pi \leq \psi < 2\pi \) belonging to the space \( L^2(SU(2)) \) such that
\[
\int_{-2\pi}^{2\pi} \int_{0}^{2\pi} \int_{0}^{\pi} |f(\phi, \theta, \psi)|^2 \sin \theta d\theta d\phi d\psi < \infty \tag{4.5}
\]
can be expanded into the mean-convergent series

\[ f(\phi, \theta, \psi) = \sum_{l} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \alpha_{mn}^l e^{-i(m\phi + n\psi)} p_{mn}^l(\cos \theta), \]  

(4.6)

where

\[ \alpha_{mn}^l = \frac{2l + 1}{16\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\phi, \theta, \psi) e^{i(m\phi + n\psi)} p_{mn}^l(\cos \theta) \sin \theta \, d\phi \, d\psi. \]  

(4.7)

In addition, we obtain from Parseval's equality that

\[ \left\{ \sum_{l} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \frac{1}{2l + 1} |\alpha_{mn}^l|^2 \right\}^{1/2} \leq \sqrt{1 + \frac{2}{n-1}} \omega_k \left( f, \frac{1}{n} \right)_2. \]  

(4.8)

Using Theorem 3.2, we obtain the following theorem.

**Theorem 4.1.** If \( f(\phi, \theta, \psi) \in L_2(SU(2)) \), then

\[ E_n(f) \leq \sqrt{1 + \frac{2}{n-1}} \omega_k \left( f, \frac{1}{n} \right)_2, \]  

(4.9)

Using the relation between the polynomial \( P_n^{(\alpha, \beta)}(z) \) and \( p_{mn}^l(z) \), we conclude that

\[ p_{mn}^l(z) = 2^{-m} \left[ \frac{(l-m)!(l+m)!}{(l-n)!(l+n)!} \right]^{1/2} (1-z)^{(m-n)/2} (1+z)^{(m+n)/2} p_{l-m}^{(m-n,m+n)}(z). \]  

(4.10)

The Jacobi polynomials obtained here are characterized by the condition that \( \alpha \) and \( \beta \) are integers and \( n + \alpha + \beta \in \mathbb{Z}_+ \).

Now, we consider the following case. Let \( L_2^{(\alpha, \beta)}[-1, 1] \) be the Hilbert space of the functions \( f \) defined on the segment \([-1, 1]\) with the scalar product

\[ (f_1, f_2) = \int_{-1}^{1} f_1(x) f_2(x) (1-x)^\alpha (1+x)^\beta \, dx; \]  

(4.11)

then, any function \( f \) in this space is expanded into the mean-convergent series

\[ f(x) = \sum_{n=0}^{\infty} \alpha_n \hat{P}_n^{(\alpha, \beta)}(x), \]  

(4.12)
where the polynomials $\hat{P}^{(\alpha,\beta)}_n(x)$ are given by

$$
\hat{P}^{(\alpha,\beta)}_k(x) = 2^{-(\alpha+\beta+1)/2} \left[ \frac{k!(k+\alpha+\beta)!(\alpha+\beta+2k+1)}{(k+\alpha)!(k+\beta)!} \right]^{1/2} P^{(\alpha,\beta)}_k(x),
$$

(4.13)

$$
\alpha_n = \int_{-1}^{1} f(x) \hat{P}^{(\alpha,\beta)}_n(x) (1-x)^\alpha (1+x)^\beta dx.
$$

(4.14)

The Parseval’s equality

$$
\int_{-1}^{1} |f(x)|^2 (1-x)^\alpha (1+x)^\beta dx = \sum_{n=0}^{\infty} |\alpha|^2
$$

(1.15)

holds. The formulas (4.12), (4.14), and (4.15) are proved for integral nonnegative values of $\alpha$ and $\beta$. We can show that they are valid for arbitrary real values of $\alpha$ and $\beta$ exceeding $-1$. Finally, we reach the following theorem.

**Theorem 4.2.** If $f(x) \in L_2[-1,1]$, then the following hold for Jacobi series:

$$
E_n(f)_2 \leq \sqrt{1 + \frac{2}{n-1}} \omega_k \left( f, \frac{1}{n} \right)_2,
$$

$$
\left\{ \sum_{l=n}^{\infty} |\alpha_l|^2 \right\}^{1/2} \leq \sqrt{1 + \frac{2}{n-1}} \omega_k \left( f, \frac{1}{n} \right)_2.
$$

(4.16)

**Note.** For the ideas similar to this paper we refer to [2] and its references.

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