ASYMPTOTIC ALMOST PERIODICITY OF \( C \)-SEMIGROUPS

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Let \( \{T(t)\}_{t\geq0} \) be a \( C \)-semigroup on a Banach space \( X \) with generator \( A \). We will investigate the asymptotic almost periodicity of \( \{T(t)\} \) via the Hille-Yosida space of its generator.

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1. Introduction. Motivated by the abstract Cauchy problem

\[
\frac{d}{dt} u(t) = Au(t) \quad (t \geq 0), \quad u(0) = x, \tag{1.1}
\]

a generalization of strongly continuous semigroups, \( C \)-semigroups, has recently received much attention (see [6, 7, 17]). The operator \( A \) generating a \( C \)-semigroup leads to (1.1) having a unique solution, whenever \( x = Cy \) for some \( y \) in the domain of \( A \). It is well known that the class of operators that generate \( C \)-semigroups is much larger than the class of operators that generate strongly continuous semigroups.

On the other hand, the asymptotic almost periodicity of \( C_0 \)-semigroup has been studied systematically in [2, 4, 8, 11, 12, 14, 15, 16]. It was shown that when \( A \) generates an asymptotically almost periodic \( C_0 \)-semigroup on a Banach space \( X \), then \( X \) can be decomposed into the direct sum of two subspaces \( X_a \) and \( X_s \), and the mild solutions with initial values taken from \( X_a \) are almost periodic and thus can be extended to the whole line while the mild solutions from \( X_s \) are vanishing at infinity.

In this paper, we will discuss the asymptotic almost periodicity of \( C \)-semigroups. We show that if \( A \) generates an asymptotically almost periodic \( C \)-semigroup, then the range of \( C \) has an analogous decomposition. The technique we use here is the Hille-Yosida space for \( A \), which is a maximal imbedded subspace of \( X \) such that the part of \( A \) on this subspace generates a \( C_0 \)-semigroup. The crucial facts are that a mild solution of the abstract Cauchy problem is asymptotically almost periodic in the Hille-Yosida space if and only if it is asymptotically almost periodic in \( X \), and the mild solution is vanishing at infinity in the Hille-Yosida space if and only if it is vanishing in \( X \).

Under suitable spectral conditions, we obtain a theorem for asymptotic almost periodicity of \( C \)-semigroups that is more easily testified (Theorem 3.7).
At last, we give a theorem for asymptotically almost periodic integrated semigroups (Theorem 3.9).

Although there are papers devoted to asymptotic properties of individual solution \([1, 2, 3, 4]\), our main concern is some kind of global property. Throughout the paper, all operators are linear. We write \(D(A)\) for the domain of an operator \(A\), \(R(A)\) for the range, and \(\rho(A)\) for the resolvent set; \(X\) will always be a Banach space, and the space of all bounded linear operators on \(X\) will be denoted by \(B(X)\) while \(C\) will always be a bounded, injective operator on \(X\). Finally, \(\mathbb{R}^+\) will be the half-line \([0, +\infty)\), and \(\mathbb{C}^+\) will be the half-plane \(\{z \mid z = \tau + iw, \tau, w \in \mathbb{R} \text{ and } \tau > 0\}\).

2. Preliminaries. We start with the definitions and properties of \(C\)-semigroups.

**DEFINITION 2.1.** A family \(\{T(t)\}_{t \geq 0} \subset B(X)\) is a \(C\)-semigroup if
(a) \(T(0) = C\),
(b) the map \(t \mapsto T(t)x\), from \([0, +\infty)\) into \(X\), is continuous for all \(x \in X\),
(c) \(CT(t+s) = T(t)T(s)\).

The **generator** of \(\{T(t)\}_{t \geq 0}\), \(A\) is defined by

\[
Ax = C^{-1}\left[\lim_{t \downarrow 0} \frac{1}{t}(T(t)x - Cx)\right]
\]

(2.1)

with

\[
D(A) = \left\{x \in X \mid \lim_{t \downarrow 0} \frac{1}{t}(T(t)x - Cx) \text{ exists and is in } R(C)\right\}.
\]

(2.2)

The complex number \(\lambda\) is in \(\rho_C(A)\), the \(C\)-resolvent set of \(A\), if \((\lambda - A)\) is injective and \(R(C) \subseteq R(\lambda - A)\). And we denote by \(\sigma_C(A)\) the set of all points in complex plane which are not in the \(C\)-resolvent of \(A\).

For the basic properties of \(C\)-semigroups and their generators, we refer to [7]. Next, we introduce the Hille-Yosida space for an operator.

**DEFINITION 2.2.** Suppose that \(A\) has no eigenvalues in \((0, \infty)\) and is a closed linear operator. The **Hille-Yosida space for \(A\)**, \(Z_0\) is defined by \(Z_0 = \{x \in X \mid \text{Cauchy problem (1.1) has a bounded uniformly continuous mild solution } u(\cdot, x)\}\) with

\[
\|x\|_{Z_0} = \sup \{\|u(t,x)\| : t \geq 0\} \text{ for } x \in Z_0.
\]

(2.3)

**LEMMA 2.3** (see [7]). **Suppose that \(A\) generates a strongly uniformly continuous bounded \(C\)-semigroup \(\{T(t)\}_{t \geq 0}\) on \(X\). Then, the Hille-Yosida space for**
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Let $A$, $Z_0$ be a Banach space and

$$Z_0 = \{ x \mid t \rightarrow C^{-1}T(t)x \text{ is uniformly continuous and bounded} \}$$  \hspace{1cm} (2.4)

with

$$\|x\|_{Z_0} = \sup \{ \|C^{-1}T(t)x\| : t \geq 0 \}. \hspace{1cm} (2.5)$$

Moreover, $Z_0 \hookrightarrow X$, that is, $Z_0$ can be continuously imbedded in $X$, $A|_{Z_0}$ generates a contraction $C_0$-semigroup $S(t) = C^{-1}T(t)$ on $Z_0$, and $T(t)x = S(t)Cx$ for all $x \in X$.

Let $J = \mathbb{R}^+$ or $\mathbb{R}$. The spaces of all bounded continuous functions from $J$ into $X$ will be denoted by $C_b(J, X)$, $C_0(\mathbb{R}^+, X)$ will designate the set of those $\varphi \in C_b(\mathbb{R}^+, X)$ that vanish at infinity on $\mathbb{R}^+$, and we will hereafter assume that each of these spaces is equipped with the supremum norm. Moreover, for $f \in C_b(J, X)$, $w \in J$, we put $f_w(t) = f(t + w) \ (t \in J)$ and let $H(f) = \{ f_w : w \in J \}$ denote the set of all translates of $f$.

**Definition 2.4.** (a) A function $f \in C(J, X)$ is said to be *almost periodic* if for every $\varepsilon > 0$, there exists $l > 0$ such that every subinterval of $J$ of length $l$ contains, at least, one $\tau$ satisfying $\| f(t + \tau) - f(t) \| \leq \varepsilon$ for $t \in J$. The space of all almost periodic functions will be denoted by $AP(J, X)$.

(b) A function $f \in C_b(\mathbb{R}^+, X)$ is said to be *asymptotically almost periodic* if for every $\varepsilon > 0$, there exist uniquely determined functions $g \in AP(\mathbb{R}, X)$ and $\varphi \in C_0(\mathbb{R}^+, X)$ such that $f = g|_{\mathbb{R}^+} + \varphi$;

(c) $H(f)$ is relatively compact in $C_b(\mathbb{R}^+, X)$.

**Lemma 2.5** (see [10, 15, 16]). *For a function $f \in C(\mathbb{R}^+, X)$, the following statements are equivalent:*

(a) $f \in AAP(\mathbb{R}^+, X)$;

(b) there exist uniquely determined functions $g \in AP(\mathbb{R}, X)$ and $\varphi \in C_0(\mathbb{R}^+, X)$ such that $f = g|_{\mathbb{R}^+} + \varphi$;

(c) $H(f)$ is relatively compact in $C_b(\mathbb{R}^+, X)$.

**Definition 2.6.** Let $\{ F(t) \}_{t \in J} \subseteq B(X)$ be a strongly continuous operator family.

(a) $F(t)$ ($t \in J$) is *almost periodic* if $F(\cdot)x$ is almost periodic for every $x \in X$.

(b) $F(t)$ ($t \in \mathbb{R}^+$) is *asymptotically almost periodic* if $F(\cdot)x$ is asymptotically almost periodic for every $x \in X$.

3. **Main results.** In order to characterize the asymptotic almost periodicity of $C$-semigroups, we need the following result.
**Lemma 3.1.** Assume that \( \{T(t)\}_{t \geq 0} \) is a \( C \)-semigroup on \( X \) generated by \( A \) and \( A \) has no eigenvalues in \((0, \infty)\). Suppose that \( T(\cdot)x : \mathbb{R}^+ \to X \) is asymptotically almost periodic for some \( x \in X \). Then,

(a) there exist \( y \in X \) and \( \varphi \in C_0(\mathbb{R}^+, X) \) such that, for all \( t \geq 0 \), \( T(t)y \in R(C) \), \( C^{-1}T(\cdot)y \in \text{AP}(\mathbb{R}^+, X) \), and

\[
T(t)x = C^{-1}T(t)y + \varphi(t); \tag{3.1}
\]

(b) if \( \{T(t)\} \) is strongly uniformly continuous and bounded, then there exist \( y, z \in Z_0 \) such that \( S(\cdot)y \in \text{AP}(\mathbb{R}^+, Z_0) \), \( S(\cdot)z \in C_0(\mathbb{R}^+, Z_0) \), and

\[
T(t)x = S(t)y + S(t)z \quad \forall \ t \geq 0, \tag{3.2}
\]

where \( Z_0 \) is the Hille-Yosida space for \( A \) and \( \{S(t)\} \) is the \( C_0 \)-semigroup generated by \( A|_{Z_0} \).

**Proof.** (a) By Lemma 2.5, there exist uniquely determined functions \( h \in \text{AP}(\mathbb{R}, X) \) and \( \varphi \in C_0(\mathbb{R}^+, X) \) such that \( T(\cdot)x = h|_{\mathbb{R}^+} + \varphi \), and, from the proof of Lemma 2.5 (see [14]), we know that there exists \( 0 < t_n \to \infty \) such that

\[
h(t) = \lim_{n \to \infty} T(t + t_n)x \quad \forall \ t \geq 0. \tag{3.3}
\]

Let \( y = h(0) \), then we have \( \lim_{n \to \infty} T(t_n)x = y \). Therefore, for each \( t \geq 0 \),

\[
Ch(t) = \lim_{n \to \infty} CT(t + t_n)x = T(t)n \lim_{n \to \infty} T(t_n)x = T(t)y, \tag{3.4}
\]

which implies \( T(t)y \in R(C) \) for all \( t \geq 0 \) and \( C^{-1}T(\cdot)y = h|_{\mathbb{R}^+} \in \text{AP}(\mathbb{R}^+, X) \), so that

\[
T(t)x = C^{-1}T(t)y + \varphi(t) \quad \forall \ t \geq 0. \tag{3.5}
\]

(b) Let \( y \) and \( \varphi(t) \) be the same as in (a). Since \( C^{-1}T(\cdot)y = h|_{\mathbb{R}^+} \in \text{AP}(\mathbb{R}^+, X) \), we have that \( C^{-1}T(t)y \) is uniformly continuous and bounded in \([0, +\infty)\), and so it follows that \( y \in Z_0 \) by Lemma 2.3. Setting \( z = Cx - y \), then obviously, \( z \in Z_0 \). Hence, \( S(t)y = C^{-1}T(t)y \) and

\[
S(t)z = C^{-1}T(t)z = C^{-1}T(t)Cx - C^{-1}T(t)y = T(t)x - C^{-1}T(t)y = \varphi(t) \tag{3.6}
\]

so that

\[
T(t)x = S(t)y + S(t)z \quad \forall \ t \geq 0. \tag{3.7}
\]

Next, we show that \( S(\cdot)y \in \text{AP}(\mathbb{R}^+, Z_0) \) and \( S(\cdot)z \in C_0(\mathbb{R}^+, Z_0) \). By the definition of almost periodicity, we know, for every \( \varepsilon > 0 \), there exists \( l > 0 \) such that every subinterval of \( \mathbb{R}^+ \) of length \( l \) contains, at least, one \( \tau \) satisfying

\[
\sup_{t \geq 0} \|S(t + \tau)y - S(t)y\| < \varepsilon, \tag{3.8}
\]
hence,

\[
\sup_{t \geq 0} \| S(t + \tau) y - S(t) y \|_{Z_0} = \sup_{t \geq 0, s \geq 0} \| C^{-1} T(s) S(t + \tau) y - C^{-1} T(s) S(t) y \| \\
= \sup_{t \geq 0, s \geq 0} \| C^{-1} T(s) C^{-1} T(t + \tau) y - C^{-1} T(s) C^{-1} T(t) y \| \\
= \sup_{t \geq 0, s \geq 0} \| C^{-1} T(t + s + \tau) y - C^{-1} T(t + s) y \| \\
= \sup_{t \geq 0} \| S(t + \tau) y - S(t) y \| < \varepsilon,
\]

which yields \( S(t) y \in \text{AP}(\mathbb{R}^+, Z_0) \). Also,

\[
\| S(t) z \|_{Z_0} = \sup_{s \geq 0} \| C^{-1} T(s) S(t) z \| = \sup_{s \geq 0} \| S(t + s) z \| = \sup_{s \geq t} \| S(s) z \| \quad (3.10)
\]

converges to 0 as \( t \to +\infty \) since \( S(t) z \in C_0(\mathbb{R}^+, X) \), and this yields \( S(t) z \in C_0(\mathbb{R}^+, Z_0) \).

Now, we can prove the following theorem which characterizes the asymptotic almost periodicity of C-semigroups.

**Theorem 3.2.** Let \( \{T(t)\} \) be a C-semigroup generated by \( A \) on \( X \). Then, \( \{T(t)\} \) is asymptotically almost periodic if and only if \( R(C) \subseteq X_{0a} + X_{0s} \), where \( X_{0a} = \{ x \mid x \in Z_0, S(t) x \in \text{AP}(\mathbb{R}^+, Z_0) \} \) and \( X_{0s} = \{ x \mid x \in Z_0, S(t) x \in C_0(\mathbb{R}^+, Z_0) \} \), where \( Z_0 \) is the Hille-Yosida space for \( A \), \( \{S(t)\} \) is the \( C_0 \)-semigroup generated by \( A|_{Z_0} \).

**Proof.** Necessity. First, it follows from [17, Lemma 1.6.(a)] and the uniform boundedness theorem that \( \{T(t)\} \) is bounded and strongly uniformly continuous. By Lemma 3.1(b), for every \( x \in X \), there exist \( y, z \in Z_0 \) such that \( S(\cdot) y \in \text{AP}(\mathbb{R}^+, Z_0), S(\cdot) z \in C_0(\mathbb{R}^+, Z_0) \), and \( T(t) x = S(t) y + S(t) z \) for all \( t \geq 0 \). Choosing \( t = 0 \), we obtain \( C x = y + z \), that is, \( R(C) \subseteq X_{0a} + X_{0s} \).

Sufficiency. Since \( R(C) \subseteq X_{0a} + X_{0s} \), for any \( x \in X \), there exist \( y \in X_{0a} \) and \( z \in X_{0s} \) such that \( C x = y + z \). Hence, by Lemma 2.3, we have \( T(t) x = S(t) C x = S(t) y + S(t) z \) while \( S(t) y \in \text{AP}(\mathbb{R}^+, Z_0) \) and \( S(t) z \in C_0(\mathbb{R}^+, Z_0) \). Since \( Z_0 \) is continuously imbedded in \( X \), it follows that \( T(t) x \) is asymptotically almost periodic by Lemma 2.5, that is, \( \{T(t)\} \) is an asymptotically almost periodic C-semigroup.

From the proof of Lemma 3.1(b), we know that for \( y, z \in Z_0 \), \( S(t) y \in \text{AP}(\mathbb{R}^+, X) \) and \( S(t) z \in C_0(\mathbb{R}^+, X) \) if and only if \( S(t) y \in \text{AP}(\mathbb{R}^+, Z_0) \) and \( S(t) z \in C_0(\mathbb{R}^+, Z_0) \), respectively. Hence, we have the following corollary.

**Corollary 3.3.** Let \( \{T(t)\}_{t \geq 0} \) be a C-semigroup generated by \( A \) on \( X \). Then, \( \{T(t)\} \) is asymptotically almost periodic if and only if for any \( x \in X \), there
exist $y, z \in Z_0$ such that $Cx = y + z$, $C^{-1}T(t)y \in \text{AP}(\mathbb{R}^+, X)$, and $C^{-1}T(t)z \in C_0(\mathbb{R}^+, X)$, where $Z_0$ is the Hille-Yosida spaces for $A$.

In the case of $\overline{R(C)} = X$, we have the following theorem.

**Theorem 3.4.** Assume that $\{T(t)\}_{t \geq 0}$ is a $C$-semigroup on $X$ and $\overline{R(C)} = X$. Then, $\{T(t)\}$ is asymptotically almost periodic if and only if $R(C) \subseteq X_a + X_s$, where $X_a = \{ x \mid x \in X, T(t)x \in \text{AP}(\mathbb{R}^+, X) \}$ and $X_s = \{ x \mid x \in X, T(t)x \in C_0(\mathbb{R}^+, X) \}$.

**Proof.** Necessity. By Lemma 3.1(a), for every $x \in X$, there exist $y \in X$ and $\varphi \in C_0(\mathbb{R}^+, X)$ such that for all $t \geq 0$, $T(t)y \in R(C)$, $C^{-1}T(t)y \in \text{AP}(\mathbb{R}^+, X)$, and

$$T(t)x = C^{-1}T(t)y + \varphi(t).$$

It follows that $T(t)Cx = CT(t)x = T(t)y + C\varphi(t)$. Setting $z = Cx - y$, then $T(t)z = T(t)Cx - T(t)y = C\varphi(t) \in C_0(\mathbb{R}^+, X)$; on the other hand, $T(\cdot)y \in \text{AP}(\mathbb{R}^+, X)$ since $C$ is bounded. So, we have $R(C) \subseteq X_a + X_s$.

 Sufficiency follows from the fact that $R(C)$ is dense in $X$ and $\text{AAP}(\mathbb{R}^+, X)$ is closed in the space of all bounded uniformaly continuous functions from $\mathbb{R}^+$ to $X$.

The following result clarifies the relations between the generator of an asymptotically almost periodic $C$-semigroup and of an asymptotically almost periodic $C_0$-semigroup.

**Theorem 3.5.** Suppose that $A$ is closed and has no eigenvalues in $(0, \infty)$, and assume that $C^{-1}AC = A$. Then, $A$ generates an asymptotically almost periodic $C$-semigroup if and only if $R(C) \subseteq Z_{\text{aap}} = (Z_0)_a + (Z_0)_s$, where $(Z_0)_a = \{ x \mid x \in X, C^{-1}ACx = 0 \}$ and $(Z_0)_s = \{ x \mid x \in X, C^{-1}ACx \text{ converges to 0 as } t \to \infty \}$ is the Cauchy problem (1.1) has an almost periodic mild solution $u(\cdot, x)$ and $(Z_0)_s$. Then, the Cauchy problem (1.1) has a mild solution $u(\cdot, x) \in C_0(\mathbb{R}^+, X)$. And $Z_{\text{aap}}$ is the maximal continuously imbedded subspace on which $A$ generates an asymptotically almost periodic $C_0$-semigroup.

**Proof.** Necessity holds by Lemma 3.1 and the relations between $C$-semigroup and solutions of the corresponding Cauchy problem [7, Theorem 3.13].

Sufficiency. By Definition 2.2, we know that both $(Z_0)_a$ and $(Z_0)_s$ are contained in $Z_0$; so $R(C) \subseteq Z_0$. Thus, by [7, Theorem 5.17] and [9, Corollary 3.14], $A$ generates a bounded $C$-semigroup; since all mild solutions with initial data taken from $R(C)$ are asymptotically almost periodic, so is the $C$-semigroup.

Now, suppose that $Y \hookrightarrow X$ and $A|_Y$ generates a contraction asymptotically almost periodic $C_0$-semigroup, then $Y \hookrightarrow Z_0$ since $Z_0$ is maximal (cf. [7, Theorem 5.5]). So, $Y \hookrightarrow Z_{\text{aap}}$ follows from the fact that the asymptotic almost periodicity of the mild solution of the abstract Cauchy problem in $Z_0$ is equivalent to the same property in $X$, and the mild solution $u(t)$ converges to 0 as $t \to \infty$ in $X$ is equivalent to $u(t) \to 0$ in $Z_0$. 


**Remark 3.6.** (a) $(Z_0)_a$ is the maximal subspace on which $A$ generates an almost periodic $C_0$-semigroup.

(b) From the proof of Lemma 3.1 and Theorems 3.2 and 3.5, it is clear that $(Z_0)_a = X_{0a}$, $(Z_0)_s = X_{0s}$.

When $\sigma_C(A) \cap i\mathbb{R}$ is countable, we obtain a result which is more easily testified for asymptotic almost periodicity of $C$-semigroups.

Let $f : \mathbb{R}^+ \to X$ be strongly measurable, and let $\tilde{f}$ be the Laplace transform of $f$,

$$\tilde{f}(z) = \int_0^{+\infty} e^{-zt} f(t) \, dt.$$ (3.12)

We assume that $\tilde{f}(z)$ exists for all $z$ in $\mathbb{C}^+$, so $\tilde{f}$ is holomorphic in $\mathbb{C}^+$ (usually, $f$ will be bounded). A point $\lambda = i\eta$ in $i\mathbb{R}$ is said to be a regular point for $\tilde{f}$ if there is an open neighborhood $U$ of $\lambda$ in $\mathbb{C}$ and a holomorphic function $g : U \to X$ such that $g(z) = \tilde{f}(z)$ whenever $z \in U \cap \mathbb{C}^+$. The singular set $E$ of $\tilde{f}$ is the set of all points of $i\mathbb{R}$ which are not regular points.

**Theorem 3.7.** Let $\{T(t)\}_{t \geq 0}$ be a $C$-semigroup generated by $A$ in $X$, and let $\sigma_C(A) \cap i\mathbb{R}$ be countable. Then, the following assertions are equivalent:

(a) $\{T(t)\}$ is asymptotically almost periodic;

(b) $\{T(t)\}$ is bounded, strongly uniformly continuous and, for every $r \in \sigma_C(A) \cap i\mathbb{R}$, $x \in X$, $\lim_{\lambda \to 0} \lambda \int_0^{+\infty} e^{-(\lambda + ir)t} T(t + s)x \, dt$ exists uniformly for $s \geq 0$.

**Proof.** (a)$\Rightarrow$(b). It follows from the properties of asymptotically almost periodic functions (cf. [4]).

(b)$\Rightarrow$(a). Given $x \in X$, let $f(t) = T(t)x$, and then we have that $f(t)$ is bounded, uniformly continuous and $\tilde{f}(\lambda) = (\lambda - A)^{-1} Cx(\text{Re} \lambda > 0)$. Let $E$ be the singular set of $\tilde{f}$ in $i\mathbb{R}$, then $E \subseteq \sigma_C(A) \cap i\mathbb{R}$, and then it follows that $E$ is countable by the assumption. Moreover, for each $ir \in \sigma_C(A) \cap i\mathbb{R}$,

$$\lim_{\lambda \to 0} \lambda \tilde{f}(\lambda + ir) = \lim_{\lambda \to 0} \int_0^{+\infty} e^{-(\lambda + ir)t} T(t + s)x \, dt$$ (3.13)

exists uniformly for $s \geq 0$, where $f_s(t) = f(s + t)$. Therefore, $f(t) = T(t)x$ is asymptotically almost periodic by [5, Theorem 4.1]; so, $\{T(t)\}$ is asymptotically almost periodic.

**Remark 3.8.** The result of Theorem 3.7 can be deduced directly from [8, Theorem 4] with the assumption on $\sigma_C(A)$ replaced by that on $\sigma(A)$; while with the aid of [5, Theorem 4.1], the result can be improved.

We end this paper with a theorem for integrated semigroups, see [13] for the definitions and basic properties of integrated semigroups.
**Theorem 3.9.** Suppose that $A$ generates a bounded $n$-times integrated semigroup \{T(t)\}_{t\geq 0} and $\sigma(A) \cap i\mathbb{R}$ is at most countable. Then, the following assertions are equivalent:

(a) \{T(t)\} is asymptotically almost periodic;

(b) for every $r \in \sigma(A) \cap i\mathbb{R}$, $x \in X$, the limit

$$
\lim_{\lambda \to 0} \int_0^{+\infty} e^{-(\lambda + ir)t} T(t + s)x \, dt
$$

exists uniformly for $s \geq 0$.

**Proof.** We only need to show (b) $\Rightarrow$ (a).

We first recall that for bounded integrated semigroup \{T(t)\}, we have

$$(\lambda - A)^{-1}x = \lambda^n \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt
$$

(3.15)

for Re$\lambda > 0$, that is,

$$
\int_0^{+\infty} e^{-\lambda t} T(t)x \, dt = \frac{1}{\lambda^n} (\lambda - A)^{-1}x
$$

(3.16)

for Re$\lambda > 0$, so that $\tilde{T}(\lambda) := \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt$ can be extended holomorphically to a connected open neighborhood $V$ of $(i\mathbb{R} \setminus \sigma(A)) \setminus \{0\}$, hence the singular set of $\tilde{T}(\lambda)$ in $i\mathbb{R}$ is contained in $(\sigma(A) \cap i\mathbb{R}) \cup \{0\}$. By our assumption and [5, Theorem 4.1], we derive (a) from (b).

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**References**


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