SOME SUBMERSIONS OF CR-HYPERSURFACES OF KAHLER-EINSTEIN MANIFOLD

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The Riemannian submersions of a CR-hypersurface $M$ of a Kaehler-Einstein manifold $\tilde{M}$ are studied. If $M$ is an extrinsic CR-hypersurface of $\tilde{M}$, then it is shown that the base space of the submersion is also a Kaehler-Einstein manifold.


1. Introduction. The study of the Riemannian submersions $\pi : M \to B$ was initiated by O’Neill [14] and Gray [9]. This theory was very much developed in the last thirty five years. Besse’s book [3, Chapter 9] is a reference work. Bejancu introduced a remarkable class of submanifolds of a Kaehler manifold that are known as CR-submanifolds (see [1, 2]). On a CR-submanifold, there are two complementary distributions $D$ and $D^\perp$, such that $D$ is $J$-invariant and $D^\perp$ is $J$-anti-invariant with respect to the complex structure $J$ of the Kaehler manifold. The integrability of the anti-invariant distribution $D$ was proved by Blair and Chen [4].

Recently, Kobayashi [10] considered the similarity between the total space of a Riemannian submersion and a CR-submanifold of a Kaehler manifold in terms of the distribution. He studied the case of a generic CR-submanifolds in a Kaehler manifold and proved that the base space is a Kaehler manifold.

In Section 3, we extend the result of Kobayashi to the general case of a CR-submanifold.

In Section 4, we study a Riemannian submersion from an extrinsic hypersurface $M$ of a Kaehler-Einstein manifold $\tilde{M}$ onto an almost-Hermitian manifold $B$. In this case, we prove that the basic manifold is a Kaehler-Einstein manifold. If $\tilde{M}$ is $C^{n+1}$, a standard example is the Hopf fibration $S^{2n+1} \to CP^n$ equipped with the canonical metrics.

For the basic formulas of Riemannian geometry, we use [11, 12].

2. Preliminaries. Let $\tilde{M}$ be a complex $m$-dimensional Kaehler manifold with complex structure $J$ and Hermitian metric $\langle \cdot, \cdot \rangle$. Bejancu [2] introduced the concept of a CR-submanifold of $\tilde{M}$ as follows: a real Riemannian manifold $M$, isometrically immersed in a Kaehler manifold $\tilde{M}$, is called a CR-submanifold of $\tilde{M}$ if there exists on $M$ a differentiable holomorphic distribution $D$ and its
orthogonal complement $D^\perp$ on $M$ is a totally real distribution, that is, $JD_{x}^\perp \subseteq T_{x}^\perp M$, where $T_{x}^\perp M$ is the normal space to $M$ at $x \in M$ for any $x \in M$. It is easily seen that each real orientable hypersurface of $M$ is a CR-submanifold. The Riemannian metric induced on $M$ will be denoted by the same symbol $\langle \cdot, \cdot \rangle$.

Let $\tilde{\nabla} \ (\text{resp., } \nabla)$ be the operator of covariant differentiation with respect to the Levi-Civita connection on $\tilde{M} \ (\text{resp., } M)$. The second-fundamental form $B$ is given by

$$B(E,F) = \tilde{\nabla}_{E}F - \nabla_{E}F$$

(2.1)

for all $E,F \in \Gamma(TM)$, where $\Gamma(TM)$ is the space of differentiable vector field on $M$. We denote everywhere by $\Gamma(\tau)$ the space of differentiable sections of a vector bundle $\tau$.

For a normal vector field $N$, that is, $N \in \Gamma(T^\perp M)$, we write

$$\tilde{\nabla}_{E}N = -L_{E}N + \nabla^\perp_{E}N,$$

(2.2)

where $-L_{E}N \ (\text{resp., } \nabla^\perp_{E}N)$ denotes the tangential (resp., normal) component of $\tilde{\nabla}_{E}N$.

Let $\mu$ be the orthogonal complementary vector bundle of $JD_{x}^\perp$ in $T_{x}^\perp M$, that is, $T_{x}^\perp M = JD_{x}^\perp \oplus \mu$.

It is clear that $\mu$ is a holomorphic subbundle of $T^\perp M$, that is, $J\mu = \mu$.

**Definition 2.1** (Kobayashi [10]). Let $M$ be a CR-submanifold of a Kaehler manifold $\tilde{M}$. A submersion from a CR-manifold $M$ onto an almost-Hermitian manifold is a Riemannian submersion $\pi : M \rightarrow M'$ with the following conditions:

(i) $D^\perp$ is the kernel of $\pi_*$,

(ii) $\pi_* : D_x \rightarrow T_{\pi(x)}M'$ is a complex isometry for every $x \in M$.

This definition is given by Kobayashi for the case where $\mu$ is a null subbundle of $T^\perp M$ (see [10]). If $JD_{x}^\perp = T_{x}^\perp M$ for any $x \in M$, we say that $M$ is a generic CR-submanifold of $\tilde{M}$ (Yano and Kon [15]). For example, any real orientable hypersurface of $\tilde{M}$ is a generic CR-submanifold of $\tilde{M}$.

Concerning the basic notions on the Riemannian submersions, see O’Neill [14] and Gray [9].

The vertical distribution of a Riemannian submersion is an integrable distribution. In our case, the distribution vertical is $D^\perp$, which is integrable according to a theorem by Blair and Chen [4].

The sections of $D^\perp$ (resp., $D$) are called the vertical vector fields (resp., the horizontal vector fields) of the Riemannian submersion $\pi : M \rightarrow M'$. The letters $U, V, W,$ and $W'$ will always denote vertical vector fields, and the letters $X, Y, Z,$ and $Z'$ denote horizontal vector fields. For any $E \in \mathfrak{X}(M)$, $vE$ and $hE$ denote the vertical and horizontal components of $E$, respectively. A horizontal vector field $X$ on $M$ is said to be basic if $X$ is $\pi$-related to a vector field $X'$ on $M'$. 
It is easy to see that every vector field $X'$ on $M'$ has a unique horizontal lift $X$ to $M$, and $X$ is basic.

Conversely, let $X$ be a horizontal vector field and suppose that $\langle X, Y \rangle_x = \langle X, Y \rangle_y$ for all $Y$ basic vector fields on $M$, for all $x, y \in \pi^{-1}(x')$, and for all $x' \in M'$. Then, the vector field $X$ is basic. We have the following O'Neill’s lemma (see [8, 14]).

**Lemma 2.2.** Let $X$ and $Y$ be basic vector fields on $M$. Then, they are satisfying the following:

(i) the horizontal component $h[X,Y]$ of $[X,Y]$ is a basic vector field and $\pi_* h[X,Y] = [X', Y'] \circ \pi$,

(ii) $h(\nabla_X Y)$ is a basic vector field corresponding to $\nabla'_X Y'$, where $\nabla'$ is the Levi-Civita connection on $(M', \langle \cdot, \cdot \rangle')$,

(iii) $[X, U] \in \Gamma(D^\perp)$ for any vertical field $U \in \Gamma(D^\perp)$.

We recall that a Riemannian submersion $\pi : (M, g) \to (M', g')$ determines the fundamental tensor field $T$ and $A$ by the formulas

$$
T_E F = h \nabla_{v_E} v F + v \nabla_{v_E} h F, \\
A_E F = v \nabla_{h_E} h F + h \nabla_{h_E} v F,
$$

(2.3)

for all $E, F \in \Gamma(TM)$ (cf. O’Neill [14] and Besse [3]).

It is easy to prove that $T$ and $A$ satisfy

$$
T_U V = T_U U, \\
A_X Y = \frac{1}{2} v [X, Y],
$$

(2.4)

(2.5)

for any $U, V \in \Gamma(D^\perp)$ and $X, Y \in \Gamma(D)$.

Formula (2.4) means that the restriction of $T$ to the integrable distribution $D^\perp$ is the second-fundamental form of the fiber submanifolds in $M$, and (2.5) measures the integrability of the distribution $D$.

We have the following properties:

$$
\nabla_U X = T_U X + h \nabla_U X, \\
\nabla_X U = v \nabla_X U + A_X U, \\
\nabla_X Y = h \nabla_X Y + A_X Y,
$$

(2.6)

for any $X, Y \in \Gamma(\mathcal{H})$ and $U \in \Gamma(\mathcal{V})$.

3. Kaehler structure on the basic space $M'$. From (2.1), we have

$$
\tilde{\nabla}_X Y = h \nabla_X Y + v \nabla_X Y + \overline{h} B(X, Y) + \overline{v} B(X, Y)
$$

(3.1)

for any $X, Y \in \Gamma(D)$. 

Here, we denote by $h$ and $v$ (resp., $\overline{h}$ and $\overline{v}$) the canonical projections on $D$ and $D^\perp$ (resp., $\mu$ and $JD^\perp$). Define a tensor field $C$ on $M$ as the vertical component $v(\nabla_X Y)$ of $\nabla_X Y$ (cf. Kobayashi [10]). The tensor field $C$ is known to be a skew-symmetric tensor field defined by Kobayashi such that

$$C(X,Y) = \frac{1}{2} v[X,Y]$$  \hspace{1cm} (3.2)

for all $X,Y \in \Gamma(D)$. Note that the tensor field $C$ is the restriction of $A$ to $\Gamma(\mathcal{J}) \times \Gamma(\mathcal{J})$.

From Definition 2.1 and Lemma 2.2, we obtain that $Jh\nabla_X Y$ (resp., $h\nabla_X JY$) is a basic vector field and corresponds to $J'\nabla'_X Y'$ (resp., $\nabla'_X J'Y'$) for any basic vector fields $X$ and $Y$ on $M$.

On the Kaehler manifold $\tilde{M}$, we have

$$\tilde{\nabla}_E JF = J\tilde{\nabla}_E F.$$  \hspace{1cm} (3.3)

From (3.1) and (3.3), we obtain the following proposition.

**Proposition 3.1.** For any basic vector fields $X$ and $Y$ on $M$,

$$Jh\nabla_X Y = h\nabla_X JY,$$  \hspace{1cm} (3.4)

$$JC(X,Y) = \overline{v}B(X,JY),$$  \hspace{1cm} (3.5)

$$C(X,JY) = J\overline{v}B(X,Y),$$  \hspace{1cm} (3.6)

$$J\overline{h}B(X,Y) = \overline{h}B(X,JY).$$  \hspace{1cm} (3.7)

**Theorem 3.2.** Let $M$ be a CR-submanifold of a Kaehler manifold $\tilde{M}$ and $\pi : M \to M'$ be a CR-submersion of $M$ on an almost-Hermitian manifold $M'$. Then, $M'$ is a Kaehler manifold.

**Proof.** From Lemma 2.2 and (3.4), we obtain that $\nabla'_{X'} J'Y' = J'\nabla'_X Y'$, so that $M'$ is a Kaehler manifold. \qed

**Remark 3.3.** Proposition 3.1 is proved for generic CR-submanifolds of $\tilde{M}$ (i.e., $\mu = 0$) in [10].

**4. Riemannian submersions from extrinsic hyperspheres of Einstein-Kaehler manifolds.** We recall that a totally umbilical submanifold $M$ of a Riemannian manifold $\tilde{M}$ is a submanifold whose first-fundamental form and second-fundamental form are proportional.

The extrinsic hyperspheres are defined to be totally umbilical hypersurfaces, having nonzero parallel mean-curvature vector field (cf. Nomizu and Yano [13]). Many of the basic results concerning extrinsic spheres in Riemannian and Kaehlerian geometry were obtained by Chen [5, 6, 7].
Let $M$ be an orientable hypersurface in a Kaehler manifold $\tilde{M}$. Then, $M$ is an extrinsic hypersphere of $\tilde{M}$ if it satisfies

$$B(E,F) = \langle E,F \rangle H$$

(4.1)

for any vector fields $E$ and $F$ on $M$. Here, $H$ denote the mean-curvature vector field of $M$. If we put $k = \|H\|$ (where the norm $\| \cdot \|$ is, with respect to a scalar product, induced on every tangent space to $M$), then $k$ is a nonzero constant function on the extrinsic hypersphere $M$.

We denote by $N$ the global unit normal vector field to $M$. Then, $\xi = -JN$ is a global unit vector on $M$ such that $N = J\xi$. Let $D$ be the maximal $J$-invariant subspace (with respect to $J$) of the tangent space $T_pM$ for every $p \in M$. We see that $M$ is a CR-hypersurface of $M$ such that $TM = D \oplus D^\perp$, where $D^\perp$ is the one-dimensional anti-invariant distribution generated by the vector field $\xi$ on $M$.

The anti-invariant distribution $D^\perp$ is integrable, and its leaves are totally geodesic in $M$ (but not in $\tilde{M}$).

This is an easy consequence from Gauss and Weingarten’s formulas of the leaves of $D^\perp$ in $M$. This means that O’Neill’s tensor $T$ vanishes on the fibres of the Riemannian submersion $\pi : M \to B$.

The main result of this section is the following theorem.

**Theorem 4.1.** Let $M$ be an orientable extrinsic hypersphere of an Kaehler-Einstein manifold $\tilde{M}$. If $\pi : M \to B$ is a CR-submersion of $M$ on an almost-Hermitian manifold $B$, then $B$ is an Kaehler-Einstein manifold.

To prove Theorem 4.1, we need several lemmas.

**Lemma 4.2.** Following the assumptions of Theorem 4.1, then

$$\langle A_x\xi, A_y\xi \rangle = k^2 \langle X,Y \rangle$$

(4.2)

for any horizontal vector $X$ on $M$.

**Proof.** From Gauss’s formula (2.1) and the umbilicality of $M$, we get $\tilde{\nabla}_X\xi = \nabla_X\xi$ for any vector field $X$ on $M$. Then, we have

$$\langle \tilde{\nabla}_XJN,Y \rangle = \langle \nabla_X\xi,Y \rangle = \langle h\nabla_X\xi,Y \rangle = \langle A_x\xi,Y \rangle.$$  

(4.3)

On the other hand, $\tilde{M}$ is a Kaehler manifold, so that $\nabla$ commute with $J$:

$$\langle \tilde{\nabla}_XJN,Y \rangle = \langle J\tilde{\nabla}_XN,Y \rangle = -\langle \tilde{\nabla}_XN,JY \rangle = \langle B(X,JY),N \rangle = \langle G(X,JY)H,N \rangle = k \langle X,JY \rangle.$$  

(4.4)
Consequently,

\[ \langle A_X \xi, A_Y \xi \rangle = k \langle X, J A_Y \xi \rangle = -k \langle J X, A_Y \xi \rangle = k^2 \langle X, Y \rangle. \]  

(4.5)

**Lemma 4.3.** Following the assumptions of Theorem 4.1, then

\[ \langle A_X Y, A_Z W \rangle = k^2 \langle X, J Y \rangle \langle Z, J W \rangle \]  

(4.6)

for any horizontal vector fields on \( M \).

**Proof.** We say that \( A_X Y \) is a vertical vector field, hence

\[ A_X Y = \langle A_X Y, \xi \rangle \xi. \]  

(4.7)

Then,

\[ \langle A_X Y, A_Z W \rangle = \langle A_X Y, \xi \rangle \langle A_Z W, \xi \rangle = k^2 \langle X, J Y \rangle \langle Z, J W \rangle. \]  

(4.8)

**Lemma 4.4.** Following the assumptions of Theorem 4.1, then

\[ \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + k^2 \{ \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle \}, \]  

(4.9)

where \( \tilde{R} \) and \( R \) are the curvature tensor on \( \tilde{M} \) and \( M \), respectively.

**Proof.** We have the Gauss equation

\[ \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + \langle B(X, Z), B(Y, W) \rangle - \langle B(Y, Z), B(X, W) \rangle. \]  

(4.10)

Using the umbilicality condition, we get (4.9).

**Lemma 4.5.** For any horizontal vector fields \( X \) and \( Y \) on \( M \),

\[ \tilde{R}(\xi, X, Y, \xi) = 0, \quad \tilde{R}(\xi, J X, J Y, \xi) = 0. \]  

(4.11)

**Proof.** For a Riemannian submersion with totally geodesic fibres, the following formula is known:

\[ \tilde{R}(X, V, Y, U) = \langle (\nabla_V A)(X, Y), U \rangle + \langle A_X V, A_Y U \rangle. \]  

(4.12)

On the other hand, the first term on the right part is skew-symmetric with respect to the vertical vector fields \( V \) and \( U \). From (4.12) and (4.9), we obtain (4.11).
Proof of Theorem 4.1. For the horizontal vector fields $X, Y, Z,$ and $W$ on $M$, we have the following equation of O’Neill:

$$R(X, Y, Z, W) = R'(X', Y' \cdot Z', W') - 2\langle A_X Y, A_Z W \rangle + \langle A_Y Z, A_X W \rangle - \langle A_X Z, A_Y W \rangle$$

(4.13)

(see [3, 14]).

By (4.9) and (4.11), we get the following formula that connects the curvature of $M'$ to the curvature of the Kaehler manifold $\tilde{M}$:

$$\tilde{R}(X, Y, Z, W) = R'(X', Y', Z', W') - k^2 \left\{ \langle X, J Z \rangle \langle Y, J W \rangle - \langle X, J W \rangle \langle Y, J Z \rangle \right\} + 2\langle X, J Y \rangle \langle Z, J W \rangle$$

$$- k^2 \left\{ \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle \right\}.$$

(4.14)

Let $(e_1, \ldots, e_p; Je_1, \ldots, J e_p)$ be a local $J$-frame of basic vector fields for the horizontal distribution $D$. Then, $(e_1, \ldots, e'_p; J'e_1, \ldots, J'e_p)$ is a local $J'$-frame if $\pi_{\text{star}} e_i = e'_i$ on the Kaehler manifold $B$.

Using the above lemmas, from (4.14) by a straightforward calculation, we conclude that $B$ is a Kaehler-Einstein manifold if $\tilde{M}$ is a Kaehler-Einstein manifold.

\[ \square \]

Corollary 4.6. Let $\tilde{M}$ be a complex-form space and $M$ an orientable CR-hypersurface of $\tilde{M}$. Then, the base space of submersion $\pi : M \to B$ is also a complex-form space.

Proof. The corollary follows by straightforward calculation making use of (4.14).

\[ \square \]

Example 4.7. Let $S^{2n+1}$ be the standard hypersphere in $C^{n+1}$. Then, $S^{2n+1}$ is an extrinsic hypersphere in $C^{n+1}$, and we have the Hopf fibration $\pi : S^{2n+1} \to CP^n$ equipped with the canonical metrics.

References


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