RIESZ BASES AND POSITIVE OPERATORS
ON HILBERT SPACE

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It is shown that a normalized Riesz basis for a Hilbert space \( H \) (i.e., the isomorphic image of an orthonormal basis in \( H \)) induces in a natural way a new, but equivalent, inner product on \( H \) in which it is an orthonormal basis, thereby extending the sense in which Riesz bases and orthonormal bases are thought of as being the same. A consequence of the method of proof of this result yields a series representation for all positive isomorphisms on a Hilbert space.

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1. Introduction. Let \( H \) denote a Hilbert space (assumed real, for notational convenience) with inner product \((\cdot, \cdot)\) and let \( \{x_i\} \) be a basis for \( H \) having coefficient functionals \( \{f_i\} \) denoted by \( \{x_i, f_i\} \). We say that \( \{x_i, f_i\} \) is a Riesz basis for \( H \) if it has the property that \( \sum a_i x_i \) converges in \( H \) if and only if \( \{a_i\} \) is in the sequence space \( l^2 \). Equivalently, \( \{x_i, f_i\} \) is a Riesz basis for \( H \) if and only if there is an isomorphism \( U \) on \( H \) and some orthonormal basis \( \{\phi_i\} \) for \( H \) so that \( U \phi_i = x_i \) for all \( i \), implying that Riesz bases and orthonormal bases are the "same" in linear-topological terms, but differ in geometrical ones due to the additional orthogonality relations between basis vectors in an orthonormal basis that is lacking in a Riesz basis. The result below (Theorem 2.1) shows that this is, in a sense, an artificial distinction by showing that every Riesz basis, in fact, is an orthonormal basis for \( H \) under a different, but equivalent, inner product.

2. Main results

**Theorem 2.1.** Let \( \{x_i, f_i\} \) be a normalized Riesz basis for a Hilbert space \( H \). Then there is an equivalent inner product on \( H \) in which \( \{x_i\} \) is an orthonormal basis for \( H \) under the norm induced by this inner product.

**Proof.** If \( x \) and \( y \) are any two vectors in \( H \), then the sequences \( \{(f_i, x)\} \) and \( \{(f_i, y)\} \) are in \( l^2 \), implying that \( \sum (f_i, x)(f_i, y) \) converges. Clearly, the bilinear form on \( H \times H \), defined by \( \langle x, y \rangle = \sum (f_i, x)(f_i, y) \), is then an inner product on \( H \) for which \( \langle x_i, x_j \rangle = d_{ij} \) for all \( i \) and \( j \), in which \( \{x_i\} \) is an orthonormal set that is also complete, since if \( \langle x_n, x \rangle = 0 \) for all \( n \), then \( 0 = \sum (f_i, x_n)(f_i, x) = (f_n, x) \) for all \( n \); that is, \( 0 = \sum (f_i, x_n)(f_i, x) \) by definition of the new inner product for all \( n \), implying that \( (f_n, x) = 0 \) for all \( n \), and hence that \( x = 0 \).
As usual, the inner product $\langle \cdot, \cdot \rangle$ defines a norm $\| \cdot \|_1$ on $H$ by $\|x\|_1^2 = \langle x, x \rangle = \sum |(f_i, x)|^2$. Since $\{x_i\}$ is a Riesz basis, there is an isomorphism $U$ on $H$ that maps each vector $\phi_i$ in an orthonormal basis $\{\phi_i\}$ for $H$ to the vector $x_i$, implying that the isomorphism $V = (U^*)^{-1}U^{-1}$ on $H$ maps $x_i$ to $f_i$ for all $i$. Since, for any $x$ in $H$, $\langle x, x \rangle = \sum (f_i, x)(f_i, x) = (\sum (f_i, x)(Vx_i, x)) = \sum (f_i, x)(Vx_i, x) = (V[\sum (f_i, x)x_i], x) = (Vx, x)$, we see that $(Vx, x) = \sum |(f_i, x)|^2 = \|x\|_1^2$ for all $x$ in $H$, so $V$ is a positive operator. If we let $W$ denote the positive square root of $V$, then $W$ is also an isomorphism on $H$ so that, for any $x$ in $H$, we have $\|x\|_1^2 = (Vx, x) = (Wx, Wx) = \|Wx\|^2 \leq \|w\|^2 \|x\|^2$. In the same way, we see that $\|x\|_1^2 \leq \|W^{-1}\|^2 \|x\|^2$, and it follows that the new norm $\| \cdot \|_1$ is equivalent to the original norm on $H$. In particular, $H$ is then complete under the new norm, hence a Hilbert space, in which $\{x_i\}$ is then an orthonormal basis, being an orthonormal set, that is complete in the new inner product.

3. Positive operators. In the proof above we used the fact that if $\{x_i, f_i\}$ is a Riesz basis for a Hilbert space $H$, then the operator $U$ on $H$, mapping $x_i$ to $f_i$, is a positive isomorphism on $H$. It is interesting to note that, in fact, every positive isomorphism on $H$ is such an operator for some Riesz basis in $H$, thereby providing a representation for all positive isomorphisms $U$ on a Hilbert space.

**Theorem 3.1.** An operator $U$ on a Hilbert space $H$ is a positive isomorphism if and only if $U$ is of the form $U = \sum f_i \otimes f_i$ for some Riesz basis $\{x_i, f_i\}$ for $H$ (i.e., $Ux_i = f_i$ for all $i$).

**Proof.** If $U = \sum f_i \otimes f_i$ for some Riesz basis $\{x_i, f_i\}$ for $H$, $\{\phi_i\}$ is an orthonormal basis for $H$, and $T$ is the isomorphism on $H$ mapping $\phi_i$ to $f_i$ for all $i$, then $U = \sum T\phi_i \otimes T\phi_i = TT^*$, a positive isomorphism on $H$.

Conversely, if $U$ is any positive isomorphism on $H$, then $W$, the positive square root of $U$, is also an isomorphism on $H$. If we set $f_i = W\phi_i$ for some orthonormal basis $\{\phi_i\}$, then $\{f_i\}$ is a Riesz basis for $H$ so that, for any $x$ in $H$, we have $Ux = W^2x = W[\sum (f_i, x)f_i] = W[\sum (W\phi_i, x)\phi_i] = \sum (f_i, x)W\phi_i = \sum (f_i, x)f_i$. That is, $U = \sum f_i \otimes f_i$ and the proof is complete. \qed

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