KY FAN INEQUALITY AND BOUNDS FOR DIFFERENCES OF MEANS

PENG GAO

Received 23 July 2002

We prove an equivalent relation between Ky Fan-type inequalities and certain bounds for the differences of means. We also generalize a result of Alzer et al. (2001).

2000 Mathematics Subject Classification: 26D15, 26D20.

1. Introduction. Let \( P_{n,r}(x) \) be the generalized weighted power means: 
\[
P_{n,r}(x) = \left( \sum_{i=1}^{n} \omega_i x_i^r \right)^{1/r},
\]
where \( \omega_i > 0 \), \( 1 \leq i \leq n \) with \( \sum_{i=1}^{n} \omega_i = 1 \) and \( x = (x_1, x_2, \ldots, x_n) \). Here, \( P_{n,0}(x) = \prod_{i=1}^{n} x_i^{\omega_i} \) denotes the limit of \( P_{n,r}(x) \) as \( r \to 0^+ \), which can be proved by noting that if \( p(r) = \ln \left( \sum_{i=1}^{n} \omega_i x_i^r \right) \), then \( p'(0) = \ln(\prod_{i=1}^{n} x_i^{\omega_i}) = \ln(P_{n,0}(x)) \). We write \( P_{n,r} \) for \( P_{n,r}(x) \) when there is no risk of confusion.

In this paper, we assume that \( 0 < x_1 \leq x_2 \leq \cdots \leq x_n \). With any given \( x \), we associate \( x' = (1 - x_1, 1 - x_2, \ldots, 1 - x_n) \) and write \( A_n = P_{n,1}, \ G_n = P_{n,0}, \) and \( H_n = P_{n,-1} \). When \( 1 - x_i \geq 0 \) for all \( i \), we define \( A'_n = P_{n,1}(x') \) and similarly for \( G'_n \) and \( H'_n \). We also let \( \sigma_n = \sum_{i=1}^{n} \omega_i [x_i - A_n]^2 \).

The following counterpart of the arithmetic mean-geometric mean inequality, due to Ky Fan, was first published by Beckenbach and Bellman [7].

**Theorem 1.1.** For \( x_i \in (0, 1/2) \),
\[
\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n}
\]  
with equality holding if and only if \( x_1 = \cdots = x_n \).

In this paper, we consider the validity of the following additive Ky Fan-type inequalities (with \( x_1 < x_n < 1 \)):
\[
\frac{x_1}{1 - x_1} < \frac{P_{n,r} - P_{n,s}}{P_{n,r} - P_{n,s}} < \frac{x_n}{1 - x_n}.
\]  
(1.2)

Note that by a change of variables \( x_i - 1 - x_i \), the left-hand side inequality is equivalent to the right-hand side inequality in (1.2). We can deduce (see [9]) **Theorem 1.1** from the case \( r = 1, s = 0 \), and \( x_n \leq 1/2 \) in (1.2), which is a result
of Alzer [5]. Gao [9] later proved the validity of (1.2) for \( r = 1, -1 \leq s < 1 \), and \( x_n \leq 1/2 \).

What is worth mentioning is a nice result of Mercer [12] who showed that the validity of \( r = 1 \) and \( s = 0 \) in (1.2) is a consequence of a result of Cartwright and Field [8] who established the validity of \( r = 1 \) and \( s = 0 \) for the following bounds for the differences between power means \( (r > s) \):

\[
\frac{r-s}{2x_1}\sigma_n \geq P_{n,r} - P_{n,s} \geq \frac{r-s}{2x_n}\sigma_n,
\]

where the constant \((r-s)/2\) is the best possible (see [10]).

We point out that inequalities (1.2) and (1.3) do not hold for all \( r > s \). We refer the reader to the survey article [2] and the references therein for an account of Ky Fan’s inequality, and to [4, 5, 10, 11] for other interesting refinements and extensions of (1.3).

Mercer’s result reveals a close relation between (1.3) and (1.2), and it is our main goal in the paper to prove that the validities of (1.3) and (1.2) are equivalent for fixed \( r \) and \( s \). As a consequence of this result, we give a characterization of the validity of (1.3) for \( r = 1 \) or \( s = 1 \). A solution of an open problem from [11] is also given.

Among the numerous sharpenings of Ky Fan’s inequality in the literature, we have the following inequalities connecting the three classical means (with \( \omega_i = 1/n \) here):

\[
\left( \frac{H_n}{H_n'} \right)^{n-1} A_n A'_n \leq \left( \frac{G_n}{G_n'} \right)^n \leq \left( \frac{A_n}{A_n'} \right)^{n-1} H_n H'_n.
\]

The right-hand side inequality of (1.4) is due to W. L. Wang and P. F. Wang [14] and the left-hand side inequality was recently proved by Alzer et al. [6].

It is natural to ask whether we can extend the above inequality to the weighted case, and using the same idea as in [6], we show that this is indeed true in Section 5.

### 2. The main theorem

**Theorem 2.1.** For fixed \( r > s \), the following inequalities are equivalent: (i) inequality (1.2) for \( x_n \leq 1/2 \); (ii) inequality (1.2); (iii) inequality (1.3).

**Proof.** (iii)⇒(ii) follows from a similar argument as given in [12], (ii)⇒(i) is trivial, so it suffices to show that (i)⇒(iii).

Fix \( r > s \) assuming that (1.2) holds for \( x_n \leq 1/2 \). Without loss of generality, we can assume that \( x_1 < x_n \). For a given \( x = (x_1, x_2, \ldots, x_n) \), let \( y = (\varepsilon x_1, \varepsilon x_2, \ldots, \varepsilon x_n) \). We can choose \( \varepsilon \) small so that \( \varepsilon x_n \leq 1/2 \). Now, applying the right-hand side inequality (1.2) for \( y \), we get

\[
x_n (P_{n,r}(x) - P_{n,s}(x)) > \frac{1 - \varepsilon x_n}{\varepsilon^2} (P_{n,r}(y') - P_{n,s}(y')).
\]

(2.1)
Let $f(\epsilon) = P_{n,r}(y') - P_{n,s}(y')$, then $f'(0) = 0$ and $f''(0) = (r-s)\sigma_n$. Thus, by letting $\epsilon$ tend to 0, it is easy to verify that the limit of the expression on the right-hand side of (2.1) is $(r-s)\sigma_n/2$. We can consider the left-hand side of (1.2) by a similar argument and this completes the proof.

3. An application of Theorem 2.1

**Lemma 3.1.** If inequality (1.3) holds for $r > s$, then $0 \leq r + s \leq 3$.

**Proof.** Let $n = 2$, and write $\omega_1 = 1 - q$, $\omega_2 = q$, $x_1 = 1$, and $x_2 = 1 + t$ with $t \geq -1$. Let

$$D(t;r,s,q) = \frac{r-s}{2} \sum_{i=1}^{2} w_i [x_i - A_2]^2 - P_{2,r} + P_{2,s}. \quad (3.1)$$

For $t \geq 0$, $D(t;r,s,q) \geq 0$ implies the validity of the left-hand side inequality of (1.3) while for $-1 \leq t \leq 0$, $D(t;r,s,q) \leq 0$ implies the validity of the right-hand side inequality of (1.3).

Using the Taylor series expansion of $D(t;r,s,q)$ around $t = 0$, it is readily seen that $D(0;r,s,q) = D^{(1)}(0;r,s,q) = D^{(2)}(0;r,s,q) = 0$. Thus, by the Lagrangian remainder term of the Taylor expansion,

$$D(t;r,s,q) = \frac{D^{(3)}(\theta t;r,s,q)}{3!} t^3. \quad (3.2)$$

with $0 < \theta < 1$.

Since

$$\lim_{t \to 0^+} D^{(3)}(\theta t;r,s,q) = D^{(3)}(0;r,s,q), \quad (3.3)$$

a necessary condition for (1.3) to hold is $D^{(3)}(0;r,s,q) \geq 0$ for $0 \leq q \leq 1$. The calculation yields

$$D^{(3)}(0;r,s,q) = (r-s)q(q-1)((3-2r-2s)q - (3-r-s)). \quad (3.4)$$

It is easy to check that this is equivalent to $0 \leq r + s \leq 3$.

**Theorem 3.2.** Let $r > s$. If $r = 1$, inequality (1.3) holds if and only if $-1 \leq s < 1$. If $s = 1$, inequality (1.3) holds if and only if $1 < r \leq 2$.

**Proof.** A result of Gao [9] shows the validity of (1.2) for $r = 1$, $-1 \leq s < 1$, $x_n \leq 1/2$, and a similar result of his [10] shows the validity of (1.2) for $s = 1$, $1 < r \leq 2$, $x_n \leq 1/2$. Thus, it follows from Theorem 2.1 that (1.3) holds for $r = 1$, $-1 \leq s < 1$, and $s = 1, 1 < r \leq 2$. This proves the “if” part of the statement, and the “only if” part follows from the previous lemma.

\[\square\]
We note here that a special case of Theorem 3.2 answers an open problem of Mercer [11], namely, we have shown that

\[
\frac{1}{x_1} \sigma_n \geq A_n - H_n \geq \frac{1}{x_n} \sigma_n. \tag{3.5}
\]

4. Two lemmas

**Lemma 4.1.** Let \(x, b, u, \) and \(v\) be real numbers with \(0 < x \leq b, u \geq 1, v \geq 0, \) and \(u + v \geq 2, \) then

\[
f(u, v, x, b) = \frac{u + v - 1}{ux + vb} + \frac{1}{x^2(u/x + v/b)} - \frac{1}{x} - \frac{u + v - 2}{b^2(u + v)^2} v(x - b) \tag{4.1}
\]

with equality holding if and only if \(x = b\) or \(v = 0\) or \(u = v = 1.\)

**Proof.** Let \(x < b, u > 1,\) and \(v > 1.\) We have

\[
f(u, v, x, b) = v(b - x) \left( -\frac{(u - 1)b + (v - 1)x}{xb + ux} \right) + \frac{(u - 1) + (v - 1)}{b^2(u + v)^2} (u/x + v/b) - 1 - \frac{u + v - 2}{b^2(u + v)^2} v(x - b) \tag{4.2}
\]

since \(b^2(u + v)^2 > (bv + ux)(bu + vx).\) Thus, we conclude that \(f(u, v, x, b) \leq 0\) for \(0 < x \leq b, u \geq 1, v \geq 0,\) and \(u + v \geq 2.\)

**Lemma 4.2.** Let \(x, a, b, u, v,\) and \(s\) be real numbers with \(0 < x \leq a \leq b, u \geq 1, v \geq 1, u + v \geq 3,\) and \(0 \leq s \leq v,\) then

\[
\frac{u + v - 1}{ux + sa + (v - s)b} + \frac{1}{x^2(u/x + s/a + (v - s)/b)} - \frac{1}{x} - \frac{u + v - 2}{b^2(u + v)^2} (s(x - a) + (v - s)(x - b)) \leq 0 \tag{4.3}
\]

with equality holding if and only if one of the following cases is true: (1) \(x = a = b;\) (2) \(s = 0\) and \(x = b;\) (3) \(s = v\) and \(x = a.\)

**Proof.** Let \(M = \{(s, a) \in \mathbb{R}^2 | 0 \leq s \leq v, x \leq a \leq b\}.\) Furthermore, we define \(H(s, a)\) as the expression on the left-hand side of (4.3), where \((s, a) \in M.\) It suffices to show that \(H(s, a) < 0.\) We denote the absolute minimum of \(H\) by \(m = (s_0, a_0).\) If \(m\) is an interior point of \(M,\) then we obtain

\[
0 = \frac{1}{s} \frac{\partial H}{\partial a} - \frac{1}{a - b} \frac{\partial H}{\partial s} \bigg|_{(s, a) = (s_0, a_0)} = \frac{b - a}{x^4 a^2 b(u/x + s/a + (v - s)/b)^2} > 0. \tag{4.4}
\]
Hence, \( m \) is a boundary point of \( M \), so we get
\[
m \in \{(s_0,x),(s_0,b),(0,a_0),(v,a_0)\}.
\]
(4.5)

Using Lemma 4.1, we obtain
\[
\begin{align*}
H(s_0,x) &= f(u+s_0,v-s_0,x,b) \leq 0, \\
H(s_0,b) &= H(0,a_0) = f(u,v,x,b) \leq 0, \\
H(v,a_0) &= f(u,v,x,a_0) - \frac{v(u+v-2)(a_0-x)(b^2-a_0^2)}{a_0^2b^2(u+v)^2} \leq 0.
\end{align*}
\]
(4.6)

Thus, we get that if \((s,a) \in M\), then \(H(s,a) \leq 0\). The conditions for equality can be easily checked using Lemma 4.1.

5. A sharpening of Ky Fan’s inequality. In this section, we prove the following theorem.

**Theorem 5.1.** For \(0 < x_1 \leq \cdots \leq x_n\), \( q = \min \{\omega_i\} \),
\[
\frac{1-2q}{2x_1^2} \sigma_n \geq (1-q) \ln A_n + q \ln H_n - \ln G_n \geq \frac{1-2q}{2x_n^2} \sigma_n,
\]
(5.1)
\[
\frac{1-2q}{2x_1^2} \sigma_n \geq \ln G_n - q \ln A_n - (1-q) \ln H_n \geq \frac{1-2q}{2x_n^2} \sigma_n
\]
(5.2)
with equality holding if and only if \( q = 1/2 \) or \( x_1 = \cdots = x_n \).

**Proof.** The proof uses the ideas in [6]. We prove the right-hand side inequality of (5.1); the proofs for other inequalities are similar. Fix \(0 < x = x_1, x_n = b\) with \( x_1 < x_n, n \geq 2 \); we define
\[
f_n(x_n,q) = (1-q) \ln A_n + q \ln H_n - \ln G_n - \frac{1-2q}{2x_n^2} \sigma_n,
\]
(5.3)
where we regard \( A_n, G_n, \) and \( H_n \) as functions of \( x_n = (x_1,\ldots,x_n) \).

We then have
\[
g_n(x_2,\ldots,x_{n-1}) := \frac{1}{\omega_1} \frac{\partial f_n}{\partial x_1} = \frac{1-q}{A_n} + \frac{qH_n}{x_1^2} - \frac{1}{x_1} - \frac{1-2q}{x_n^2} (x_1 - A_n).
\]
(5.4)
We want to show that \( g_n \leq 0 \). Let \( D = \{(x_2,\ldots,x_{n-1}) \in R^{n-2} | 0 < x \leq x_2 \leq \cdots \leq x_{n-1} \leq b\} \). Let \( a = (a_2,\ldots,a_{n-1}) \in D \) be the point in which the absolute minimum of \( g_n \) is reached. Next, we show that
\[
a = (x,\ldots,x,a,\ldots,a,b,\ldots,b) \quad \text{with} \quad x < a < b,
\]
(5.5)
where the numbers \( x, a, \) and \( b \) appear \( r, s, \) and \( t \) times, respectively, with \( r, s, t \geq 0 \) and \( r+s+t = n-2 \).
Suppose not, this implies that two components of \( \mathbf{a} \) have different values and are interior points of \( D \). We denote these values by \( a_k \) and \( a_l \). Partial differentiation leads to

\[
\frac{B}{a_i^2} + C = 0
\]  

for \( i = k, l \), where

\[
B = q \frac{H_n^2}{x_1^2}, \quad C = -\frac{1-q}{A_n^2} + \frac{1-2q}{x_n^2}.
\]  

Since \( z \to B/z^2 + C \) is strictly monotonic for \( z > 0 \), then (5.6) yields \( a_k = a_l \). This contradicts our assumption that \( a_k \neq a_l \). Thus, (5.5) is valid and it suffices to show that \( g_n \leq 0 \) for the case \( n = 2, 3 \).

When \( n = 2 \), by setting \( x_1 = x, x_2 = b, \omega_1/q = u, \) and \( \omega_2/q = v \), we can identify \( g_2 \) as (4.1), and the result follows from Lemma 4.1.

When \( n = 3 \), by setting \( x_1 = x, x_2 = a, x_3 = b, \omega_1/q = u, \omega_2/q = s, \) and \( \omega_3/q = v - s \), we can identify \( g_3 \) as (4.3), and the result follows from Lemma 4.2.

Thus, we have shown that \( g_n = (1/\omega_1)\partial f_n/\partial x_1 \leq 0 \) with equality holding if and only if \( n = 1 \) or \( n = 2, q = 1/2 \). By letting \( x_1 \) tend to \( x_2 \), we have

\[
f_n(x_n, q) \geq f_{n-1}(x_{n-1}, q) \geq f_{n-1}(x_{n-1}, q'),
\]  

where \( x_{n-1} = (x_2, \ldots, x_n) \) with weights \( \omega_1 + \omega_2, \ldots, \omega_{n-1}, \omega_n \) and \( q' = \min\{\omega_1 + \omega_2, \ldots, \omega_n\} \). Here, we have used the following inequality, which is a consequence of (3.5) (see [9]):

\[
\ln A_n - \ln H_n \geq \frac{1}{x_n^2} \sigma_n.
\]  

It then follows by induction that \( f_n \geq f_{n-1} \geq \cdots \geq f_2 = 0 \) when \( q = 1/2 \) in \( f_2 \) or else \( f_n \geq f_{n-1} \geq \cdots \geq f_1 = 0 \), and this completes the proof.

We note that the above theorem gives a sharpening of Sierpiński’s inequality [13], originally stated for the unweighted case (\( \omega_i = 1/n \)) as

\[
H_n^{n-1} A_n \leq G_n \leq A_n^{n-1} H_n.
\]  

The following corollary gives refinements of (1.4).
Corollary 5.2. For $0 < x_1 \leq \cdots \leq x_n < 1$, $q = \min \{\omega_i\}$,

$$\left( \frac{A_n^{(1-q)} H_n^q}{G_n} \right)^{(1-x_1)^2/x_1^2} \geq \frac{A_n^{1-q} H_n^q}{G_n} \geq \left( \frac{A_n^{(1-q)} H_n^q}{G_n} \right)^{(1-x_n)^2/x_n^2} \geq 1,$$

$$\left( \frac{G_n'}{A_n^{q} H_n^{(1-q)}} \right)^{(1-x_1)^2/x_1^2} \geq \frac{G_n'}{A_n^{q} H_n^{(1-q)}} \geq \left( \frac{G_n'}{A_n^{q} H_n^{(1-q)}} \right)^{(1-x_n)^2/x_n^2} \geq 1,$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$ or $q = 1/2$.

Proof. This is a direct consequence of Theorem 5.1, following from a similar argument as in [12].

6. Concluding remarks. We note that if for $x_n \leq 1/2$, we have

$$\left( \frac{x_1}{1-x_1} \right)^{\beta} \leq \frac{p_{n,r} - p_{n,s}}{p_{n,r} - p_{n,s}} \leq \left( \frac{x_n}{1-x_n} \right)^{\alpha},$$

then $\beta \geq 1$ and $\alpha \leq 1$; otherwise, by letting $\epsilon$ tend to 0 in (2.1), we get contradictions.

It was conjectured that an additive companion of (1.4) is true (see [1])

$$n(G_n - G_n') \leq (n - 1)(A_n - A_n') + H_n - H_n'.$$

(6.2)

In [3], Alzer asked if the above conjecture is true and whether there exists a weighted version. Based on what we have got in this paper, it is natural to give the following conjecture of the weighed version of (6.2).

Conjecture 6.1. For $0 < x_1 \leq \cdots \leq x_n \leq 1/2$ and $q = \min \{\omega_i\}$,

$$G_n - G_n' \leq (1 - q)(A_n - A_n') + q(H_n - H_n').$$

(6.3)

Recently, Alzer et al. [6] asked the following question: what is the largest number $\alpha = \alpha(n)$ and what is the smallest number $\beta = \beta(n)$ such that

$$\alpha(A_n - A_n') + (1 - \alpha)(H_n - H_n') \leq G_n - G_n' \leq \beta(A_n - A_n') + (1 - \beta)(H_n - H_n')$$

(6.4)

for all $x_i \in (0, 1/2]$ $(i = 1, \ldots, n)$?

We note here that $\alpha \leq 0$ since the left-hand side inequality above can be written as

$$\alpha A_n + (1 - \alpha)H_n - G_n \leq \alpha A_n' + (1 - \alpha)H_n' - G_n'.$$
By a similar argument as in the proof of Theorem 2.1, replacing \((x_1, \ldots, x_n)\) by \((\epsilon x_1, \ldots, \epsilon x_n)\) and letting \(\epsilon\) tend to 0 in (6.5), we find that (6.5) implies that
\[
\alpha A_n + (1 - \alpha)H_n - G_n \leq 0
\] (6.6)
for any \(x\). If we further let \(x_1\) tend to 0 in (6.6), we get
\[
\alpha A_n \leq 0
\] (6.7)
which implies that \(\alpha \leq 0\).

**Acknowledgment.** The author is grateful to the referees for their helpful comments and suggestions.

**References**


Peng Gao: Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

E-mail address:penggao@umich.edu