WAVELET ANALYSIS ON A BOEHMIAN SPACE

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We extend the wavelet transform to the space of periodic Boehmians and discuss some of its properties.

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1. Introduction. The concept of Boehmians was introduced by J. Mikusiński and P. Mikusinski [7], and the space of Boehmians with two notions of convergences was well established in [8]. Many integral transforms have been extended to the context of Boehmian spaces, for example, Fourier transform [9, 10, 11], Laplace transform [13, 17], Radon transform [14], and Hilbert transform [3, 5].

On the other hand, the theory of wavelet transform is recently developed, and it has various applications in signal processing, especially to analyze non-stationary signals by providing the time-frequency representation of the signal. For a fixed $g \in L^2(\mathbb{R})$, called a mother wavelet, the wavelet transform $\Phi_g : L^2(\mathbb{R}) \to L^2(\mathbb{R} \times \mathbb{R}^+)$ is defined by

$$\Phi_g(f)(a,b) = \int_{-\infty}^{\infty} f(x) g_{a,b}(x) \, dx \quad \text{for } a > 0, \ b \in \mathbb{R}, \tag{1.1}$$

where $g_{a,b}(x) = (1/\sqrt{a}) g((x-b)/a)$, $x \in \mathbb{R}$, are called wavelets. For more details, we refer the reader to [6]. In [4], we extended the wavelet transform to a Boehmian space which properly contains $L^2(\mathbb{R})$ and studied its properties.

Holschneider [2] introduced the wavelet transform on the space $C^\infty(\mathbb{T})$ of smooth functions on the unit circle $\mathbb{T}$ of the complex plane and gave an extension to the space of periodic distributions. In Section 2, we fix some notations and discuss the theory of wavelet transform on $C^\infty(\mathbb{T})$. In Section 3, we briefly recall the periodic Boehmians, construct a new Boehmian space $\mathcal{B}(\mathcal{F}(\mathbb{Y}),(C^\infty(\mathbb{T}),*,\circ,\Delta))$, and verify some auxiliary results. In Section 4, we define wavelet transform on the space of periodic Boehmians and prove that it is consistent with the wavelet transform on $C^\infty(\mathbb{T})$. Further, we establish that the extended wavelet transform is linear and continuous with respect to $\delta$-convergence as well as $\Delta$-convergence.
2. Preliminaries. The space $C^\infty(\mathbb{T})$ consists of infinitely differentiable, periodic functions on $\mathbb{R}$ of period $2\pi$, with the Fréchet space topology induced by the increasing sequence of seminorms

$$\|\phi\|_{C^\infty(\mathbb{T})} = \sum_{n=0}^{\infty} \sup_{t \in [0,2\pi]} |\partial^p \phi(t)|. \tag{2.1}$$

We know that

$$C^\infty(\mathbb{T}) = C^\infty_+(\mathbb{T}) \oplus C^\infty_-(\mathbb{T}) \oplus K(\mathbb{T}), \tag{2.2}$$

where $C^\infty_+(\mathbb{T})$ and $C^\infty_-(\mathbb{T})$ are the subspaces consisting of functions with positive and negative Fourier coefficients, respectively, and $K(\mathbb{T})$ is the space of constant functions.

Let $\mathcal{S}(\mathbb{R})$ denote the space of rapidly decreasing functions on $\mathbb{R}$. (See [1].) Given $f \in \mathcal{S}(\mathbb{R})$, $b \in [0,2\pi]$, and $a > 0$, define $f_a, f_{b,a} \in C^\infty(\mathbb{T})$ by

$$f_a(x) = \sum_{n \in \mathbb{Z}} \frac{1}{a} f \left( \frac{x + 2n\pi}{a} \right), \quad x \in [0,2\pi],$$

$$f_{b,a}(x) = f_a(x - b), \quad x \in [0,2\pi]. \tag{2.3}$$

Let $\mathcal{S}(\mathbb{R} \times \mathbb{R}^+)$ denote the Fréchet space of all smooth functions $\eta(b,a)$ of rapid descent on $\mathbb{R} \times \mathbb{R}^+$ which are periodic functions in the variable $b$ of period $2\pi$, with the following directed family of seminorms:

$$\|\eta\|_{\mathcal{S}(\mathbb{R} \times \mathbb{R}^+)} = \sum_{0 \leq p \leq n} \sup_{a > 0} \sup_{b \in [0,2\pi]} \left| a^p \partial^l_a \partial^k_b \eta(b,a) \right|. \tag{2.4}$$

We choose a mother wavelet $g \in \mathcal{S}(\mathbb{R})$ with all moments $\int_{-\infty}^{\infty} x^n g(x)dx$ are equal to zero.

**Definition 2.1.** The wavelet transform $T_g : C^\infty(\mathbb{T}) \to \mathcal{S}(\mathbb{R} \times \mathbb{R}^+)$ is defined by

$$T_g(\phi) = \int_0^{2\pi} \phi(x) g_{b,a}(x) dx, \quad b \in \mathbb{R}, \ a > 0. \tag{2.5}$$

**Theorem 2.2.** The wavelet transform $T_g : C^\infty(\mathbb{T}) \to \mathcal{S}(\mathbb{R} \times \mathbb{R}^+)$ is continuous and linear.

**Definition 2.3.** The map $R_g : \mathcal{S}(\mathbb{R} \times \mathbb{R}^+) \to C^\infty(\mathbb{T})$ is defined by

$$(R_g \eta)(x) = \int_0^{2\pi} \int_0^\infty g_{b,a}(x) \eta(b,a) \frac{dadb}{a}. \tag{2.6}$$
**Theorem 2.4.** The map $R_\beta : \mathcal{S}(\mathbb{T}) \to C^\infty(\mathbb{T})$ is continuous and linear.

A partial inversion formula is given by the following theorem.

**Theorem 2.5.** If $\hat{g}$ is the Fourier transform of $g$ and $C^+_\beta = \int_0^\infty |\hat{g}(a)|^2 \, (da / a)$, $C^-_\beta = \int_0^\infty |\hat{g}(-a)|^2 \, (da / a)$, then

$$R_\beta \circ T_g \phi = C^+_\beta \phi, \quad \forall \phi \in C^\infty_+(\mathbb{T}),$$

$$R_\beta \circ T_g \phi = C^-_\beta \phi, \quad \forall \phi \in C^\infty_-(\mathbb{T}).$$

(2.7)

### 3. Boehmian spaces.

The triplet $(C^\infty(\mathbb{T}), \ast, \Delta)$, where $\ast : C^\infty(\mathbb{T}) \times C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ is defined by

$$(\phi \ast \psi)(x) = \int_0^{2\pi} \phi(x-t) \psi(t) \, dt, \quad x \in [0, 2\pi]$$

(3.1)

and $\Delta$ is the collection of all sequences $(\delta_k)$ from $C^\infty(\mathbb{T})$ satisfying

1. $\int_0^{2\pi} \delta_k(t) \, dt = 1$ for all $k \in \mathbb{N}$,
2. $\int_0^{2\pi} |\delta_k(t)| \, dt \leq M$ for all $k \in \mathbb{N}$, for some $M > 0$,
3. $s(\delta_k) \to 0$ as $n \to \infty$ where $s(\delta_k) = \sup \{t \in [0, 2\pi] : \delta_k(t) \neq 0\}$,

is the collection of all equivalence classes $[\phi_k / \delta_k]$ given by the equivalence relation $\sim$ defined by

$$((\phi_k), (\delta_k)) \sim ((\psi_k), (\epsilon_k)) \text{ if } \phi_k \ast \epsilon_j = \psi_j \ast \delta_k \quad \forall k,j \in \mathbb{N}$$

(3.2)

on the collection $\mathcal{A}$ of pair of sequences $((\phi_k), (\delta_k))$, $\phi_n \in C^\infty(\mathbb{T})$, $\delta_k \in \Delta$ satisfying

$$\phi_k \ast \delta_j = \phi_j \ast \delta_k, \quad \forall k,j \in \mathbb{N}.$$  

(3.3)

This triplet with addition and scalar multiplication, defined by

$$\begin{bmatrix} \phi_k \\ \delta_k \end{bmatrix} + \begin{bmatrix} \psi_k \\ \epsilon_k \end{bmatrix} = \begin{bmatrix} \phi_k \ast \epsilon_k + \psi_k \ast \delta_k \\ \delta_k \ast \epsilon_k \end{bmatrix},$$

$$\alpha \begin{bmatrix} \phi_k \\ \delta_k \end{bmatrix} = \begin{bmatrix} \alpha \phi_k \\ \delta_k \end{bmatrix},$$

(3.4)

is called the periodic Boehmian space $[15, 16]$, and we denote it by $\mathcal{B}_\mathbb{T}$.

**Definition 3.1** ($\delta$-convergence). A sequence $(x_n)$ $\delta$-converges to $x$ in $\mathcal{B}_\mathbb{T}$, denoted by $x_n \xrightarrow{\delta} x$ as $n \to \infty$ in $\mathcal{B}_\mathbb{T}$ if there exists $(\delta_k) \in \Delta$ such that
\(x_n \ast \delta_k, x \ast \delta_k \in C^\infty(\mathbb{T})\), and for each \(k \in \mathbb{N}\),

\[x_n \ast \delta_k \rightharpoonup x \ast \delta_k \quad \text{as} \quad n \to \infty \quad \text{in} \quad C^\infty(\mathbb{T}). \quad (3.5)\]

The following theorem is proved in [8].

**Theorem 3.2.** Let \(x_n, x \in \mathcal{H}_{\mathbb{T}}\), \(n \in \mathbb{N}\). \(x_n \delta \rightharpoonup x\) as \(n \to \infty \) in \(\mathcal{H}_{\mathbb{T}}\) if and only if there exist \(\phi_{n,k}, \phi_k \in C^\infty(\mathbb{T})\) such that \(x_n = [\phi_{n,k}/\delta_k], [\phi_k/\delta_k]\) and, for each \(k \in \mathbb{N}\),

\[\phi_{n,k} \rightharpoonup \phi_k \quad \text{as} \quad n \to \infty \quad \text{in} \quad C^\infty(\mathbb{T}). \quad (3.6)\]

**Definition 3.3 (\(\Delta\)-convergence).** A sequence \((x_n)\) \(\Delta\)-converges to \(x\) in \(\mathcal{H}_{\mathbb{T}}\), denoted by \(x_n \Delta \rightharpoonup x\) as \(n \to \infty \) in \(\mathcal{H}_{\mathbb{T}}\) if there exists a delta-sequence \((\delta_n)\) such that \((x_n - x) \ast \delta_n \in C^\infty(\mathbb{T})\) for each \(n \in \mathbb{N}\) and

\[(x_n - x) \ast \delta_n \rightharpoonup 0 \quad \text{as} \quad n \to \infty \quad \text{in} \quad C^\infty(\mathbb{T}). \quad (3.7)\]

Now, we construct a new Boehmian space as follows.

As in the context of Boehmian space defined in [12], we take the vector space \(\Gamma\) and the commutative semi-group as \(\mathcal{H}_{\mathbb{T}}(\mathbb{Y})\) and \((C^\infty(\mathbb{T}), \ast)\), respectively.

**Definition 3.4.** Given \(\eta \in \mathcal{H}_{\mathbb{T}}(\mathbb{Y})\) and \(\phi \in C^\infty(\mathbb{T})\), define

\[(\eta \otimes \phi)(b,a) = \int_0^{2\pi} \eta(b-t,a) \phi(t) dt. \quad (3.8)\]

**Lemma 3.5.** If \(\eta \in \mathcal{H}_{\mathbb{T}}(\mathbb{Y})\) and \(\phi \in C^\infty(\mathbb{T})\), then \(\eta \otimes \phi \in \mathcal{H}_{\mathbb{T}}\).

**Proof.** To prove that \((\eta \otimes \phi)(b,a)\) is infinitely differentiable, we show that

\[\partial_a(\eta \otimes \phi)(b,a) = (\partial_a \eta \otimes \phi)(b,a), \]

\[\partial_b(\eta \otimes \phi)(b,a) = (\partial_b \eta \otimes \phi)(b,a). \quad (3.9)\]

Fix \(a_0 > 0, b_0 \in \mathbb{R}\) arbitrarily.

Consider \(((\eta \otimes \phi)(b_0,a) - (\eta \otimes \phi)(b_0,a_0))/(a-a_0) = \int_0^{2\pi} (\eta(b_0-t,a) - \eta(b_0-t,a_0))/(a-a_0) \phi(t) dt\). Using the mean-value theorem (in the variable \(a\)), we get that the integrand is dominated by \(\|\eta\|_{\mathcal{H}(\mathbb{Y},0,1,0,0)} \|\phi\|_{C^\infty(\mathbb{T},0)}\). Therefore, we can apply Lebesgue dominated convergence theorem [18], and we get

\[
\partial_a(\eta \otimes \phi)(b_0,a_0) = \lim_{a \to a_0} \int_0^{2\pi} \frac{\eta(b_0-t,a) - \eta(b_0-t,a_0)}{a-a_0} \phi(t) dt \\
= \int_0^{2\pi} \lim_{a \to a_0} \frac{\eta(b_0-t,a) - \eta(b_0-t,a_0)}{a-a_0} \phi(t) dt \\
= \int_0^{2\pi} \partial_a \eta(b_0-t,a_0) \phi(t) dt \\
= (\partial_a \eta \otimes \phi)(b_0,a_0). \quad (3.10)
\]
By a similar argument, we can prove that \( \partial_b (\eta \circ \phi) (b, a) = (\partial_b \eta \circ \phi) (b, a) \).

Finally by a routine manipulation, we get

\[
\| \eta \circ \phi \|_{\mathcal{S}(\mathbb{T}; n, \alpha, \beta)} \leq \| \phi \|_{\mathcal{S}(\mathbb{T}; n, \alpha, \beta)},
\]

(3.11)

where \( \| \phi \|_{\mathcal{S}(\mathbb{T}; n, \alpha, \beta)} = \int_0^{2\pi} |\phi(t)| dt \). Hence, \( \eta \circ \phi \in \mathcal{S}(\mathbb{T}) \).

**Lemma 3.6.** If \( \eta \in \mathcal{S}(\mathbb{T}) \) and \( (\delta_n) \in \Delta \), then \( \eta \circ \delta_n \to \phi \) as \( n \to \infty \) in \( \mathcal{S}(\mathbb{T}) \).

**Proof.** Let \( p, k, l \in \mathbb{N}_0 \) be arbitrary. Using the mean-value theorem and a property of \( \delta \)-sequence, we get

\[
| a^p \partial_a^l \partial_b^k (\eta \circ \delta_n - \eta) (b, a) | = | a^p ((\partial_a^l \partial_b^k \eta) \circ \delta_n) (b, a) - a^p \partial_a^l \partial_b^k \eta (b, a) | \\
\leq \int_0^{2\pi} | a^p (\partial_a^l \partial_b^k \eta (b - t, a) - \partial_a^l \partial_b^k \eta (b, a)) \delta_n (t) | dt \\
\leq \| \eta \|_{\mathcal{S}(\mathbb{T}; p, l, k+1)} \int_0^{2\pi} | t | | \delta_n (t) | dt \\
\leq Ms (\delta_n) \| \eta \|_{\mathcal{S}(\mathbb{T}; p, l, k+1)},
\]

(3.12)

which tends to 0 as \( n \to \infty \). This completes the proof of the lemma.

**Lemma 3.7.** If \( \eta_n \to \eta \) as \( n \to \infty \) in \( \mathcal{S}(\mathbb{T}) \) and \( \psi \in C^\infty (\mathbb{T}) \), then \( \eta_n \circ \psi \to \eta \circ \psi \) as \( n \to \infty \).

**Proof.** Let \( p, k, l \in \mathbb{N}_0 \) be arbitrary. Now,

\[
| a^p \partial_a^l \partial_b^k (\eta_n \circ \psi - \eta \circ \psi) (b, a) | = | a^p ((\partial_a^l \partial_b^k \eta_n - \partial_a^l \partial_b^k \eta) \circ \psi) (b, a) | \\
\leq \int_0^{2\pi} | a^p (\partial_a^l \partial_b^k (\eta_n - \eta) (b, a)) \| \psi | dt \\
\leq \| \psi \|_{\mathcal{S}(\mathbb{T}; p, l, k)} \| \eta_n - \eta \|_{\mathcal{S}(\mathbb{T}; p, l, k)} \to 0 \text{ as } n \to \infty.
\]

(3.13)

Hence, the lemma follows.

**Lemma 3.8.** If \( \eta_n \to \eta \) as \( n \to \infty \) in \( \mathcal{S}(\mathbb{T}) \) and \( \delta_n \in \Delta \), then \( \eta_n \circ \delta_n \to \eta \) as \( n \to \infty \).

**Proof.** Since we have \( \eta_n \circ \delta_n \to \eta = \eta_n \circ \delta_n - \eta \circ \delta_n + \eta \circ \delta_n - \eta \) and **Lemma 3.6**, we merely prove that \( \eta_n \circ \delta_n - \eta \circ \delta_n \to 0 \) as \( n \to \infty \).

If \( p, k, l \in \mathbb{N}_0 \), then, using a property of \( \delta \)-sequence, we get

\[
| a^p \partial_a^l \partial_b^k (\eta_n - \eta) \circ \delta_n (b, a) | \\
\leq \| \eta_n - \eta \|_{\mathcal{S}(\mathbb{T}; p, l, k)} \int_0^{2\pi} | \delta_n (t) | dt \leq M | \eta_n - \eta |_{\mathcal{S}(\mathbb{T}; p, l, k)}.
\]

(3.14)

The above inequalities prove the lemma.
Now using the above lemmas we can construct the Boehmian space $\mathcal{B}_\mathcal{Y} = (\mathcal{S}_\mathcal{Y}, (C^\infty, \ast), \odot, \Delta)$ in a canonical way.

### 4. Generalized wavelet transform

**Definition 4.1.** Define $\mathcal{F}_g : \mathcal{B}_T \to \mathcal{B}_\mathcal{Y}$ by

$$\mathcal{F}_g \left( \begin{bmatrix} \phi_n \\ \delta_n \end{bmatrix} \right) = \begin{bmatrix} T_g \phi_n \\ \delta_n \end{bmatrix}.$$  \hfill (4.1)

**Theorem 4.2.** The generalized wavelet transform $\mathcal{F}_g : \mathcal{B}_T \to \mathcal{B}_\mathcal{Y}$ is well defined.

First, we state and prove a lemma that will be useful.

**Lemma 4.3.** If $\phi, \psi \in C^\infty(\mathbb{T})$, then $T_g(\phi \ast \psi) = T_g \phi \odot \psi$.

**Proof.** Let $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ be arbitrary. Now

$$T_g(\phi \ast \psi)(b,a) = \int_0^{2\pi} (\phi \ast \psi)(x) g_a(x-b) dx \hfill (4.2)$$

By an easy verification, we can apply Fubini’s theorem and the last integral equals

$$\int_0^{2\pi} \psi(t) dt \int_0^{2\pi} \phi(x-t) g_a(x-b) dx \hfill (4.3)$$

$$= \int_0^{2\pi} \psi(t) dt \int_0^{2\pi} \phi(x) g_a(x-(b-t)) dx$$

$$= (T_g \phi \odot \psi)(b,a).$$

**Proof of Theorem 4.2.** First, we show that $((T_g \phi_n), (\delta_n))$ is a quotient. Since $[\phi_n/\delta_n] \in \mathcal{B}_T$, we have

$$\phi_k \ast \delta_j = \phi_j \ast \delta_k, \quad \forall j, k \in \mathbb{N}. \hfill (4.4)$$

Applying the classical wavelet transform $T_g$ on both sides, we get

$$T_g \phi_k \odot \delta_j = T_g \phi_j \odot \phi_k, \quad \forall j, k \in \mathbb{N} \text{ (by Lemma 4.3)}. \hfill (4.5)$$

Next, we show that the definition of $\mathcal{F}_g$ is independent of the choice of the representative.
Let \( \phi_k/\epsilon_k = [\psi_k/\delta_k] \) in \( \mathcal{B}_T \). Then, we have

\[
\phi_k \ast \epsilon_j = \psi_j \ast \delta_k, \quad \forall j, k \in \mathbb{N}.
\]  

(4.6)

Again, applying the wavelet transform and using Lemma 4.3, we get

\[
\mathcal{T}_g \phi_k \odot \epsilon_j = \mathcal{T}_g \psi_j \odot \delta_k, \quad \forall j, k \in \mathbb{N}.
\]  

(4.7)

Hence, the theorem follows.

\[ \square \]

**Theorem 4.4** (consistency). Let \( \mathcal{I}_T : C^\infty (\mathbb{T}) \to \mathcal{B}_T \) and \( \mathcal{I}_Y : \mathcal{I}(\mathbb{Y}) \to \mathcal{B}_Y \) be the canonical identification defined, respectively, by

\[
\phi \mapsto \left[ \frac{\phi \ast \delta_n}{\delta_n} \right], \quad \eta \mapsto \left[ \frac{\eta \odot \delta_n}{\delta_n} \right],
\]  

(4.8)

where \( (\delta_n) \in \Delta \), then \( \mathcal{T}_g \circ \mathcal{I}_T = \mathcal{I}_Y \circ T_g \).

**Proof.** Let \( \phi \in C^\infty (\mathbb{T}) \), then

\[
\mathcal{T}_g (\mathcal{I}_T (\phi)) = \mathcal{T}_g \left( \left[ \frac{\phi \ast \delta_n}{\delta_n} \right] \right) = \left[ \frac{T_g(\phi \ast \delta_n)}{\delta_n} \right]
\]  

(by Lemma 4.3)

\[
= \mathcal{I}_Y (T_g (\phi)).
\]  

(4.9)

**Theorem 4.5.** The wavelet transform \( \mathcal{T}_g : \mathcal{B}_T \to \mathcal{B}_Y \) is a linear map.

**Proof.** If \([\phi_n/\delta_n], [\psi_n/\epsilon_n] \in \mathcal{B}_T\), then

\[
\mathcal{T}_g \left( \left[ \frac{\phi_n}{\delta_n} \right] + \left[ \frac{\psi_n}{\epsilon_n} \right] \right) = \mathcal{T}_g \left( \left[ \frac{\phi_n \ast \epsilon_n + \psi_n \ast \delta_n}{\delta_n \ast \epsilon_n} \right] \right) = \left[ \frac{T_g(\phi_n) + T_g(\psi_n)}{\delta_n} \right]
\]  

(4.10)

If \( \alpha \in \mathbb{C} \) and \([\phi_n/\delta_n] \in \mathcal{B}_T\), then

\[
\mathcal{T}_g \left( \alpha \left[ \frac{\phi_n}{\delta_n} \right] \right) = \mathcal{T}_g \left( \left[ \frac{\alpha \phi_n}{\delta_n} \right] \right) = \left[ \frac{T_g(\alpha \phi_n)}{\delta_n} \right] = \alpha \left[ \frac{T_g \phi_n}{\delta_n} \right]
\]  

(4.11)

In the above proof, we have used the fact that \( T_g \) is linear wherever it is required.

\[ \square \]
From the following two theorems, we say that the generalized wavelet transform is continuous with respect to $\delta$-convergence as well as $\Delta$-convergence.

**Theorem 4.6.** If $x_n \xrightarrow{\delta} x$ as $n \to \infty$ in $\mathcal{B}_Y$, then $\mathcal{T}_g x_n \xrightarrow{\delta} \mathcal{T}_g x$ as $n \to \infty$ in $\mathcal{B}_Y$.

**Proof.** If $x_n \xrightarrow{\delta} x$ as $n \to \infty$, then, by Theorem 3.2, there exist $\phi_{n,k}, \phi_k \in C^\infty(T)$ and $(\delta_k) \in \Delta$ such that $x_n = [\phi_{n,k}/\delta_k]$ and $x = [\phi_k/\delta_k]$ and, for each $k \in \mathbb{N}$, $\phi_{n,k} \to \phi_k$ as $n \to \infty$ in $C^\infty(T)$.

By the continuity of the classical wavelet transform, we have, for each $k \in \mathbb{N}$,

$$T_g \phi_{n,k} \to T_g \phi_k \quad \text{as} \quad n \to \infty \quad \text{in} \quad \mathcal{H}_2 Y. \quad (4.12)$$

Since $\mathcal{T}_g (x_n) = [T_g \phi_{n,k}/\delta_k]$ and $\mathcal{T}_g (x) = [T_g \phi_k/\delta_k]$, we get $\mathcal{T}_g (x_n) \xrightarrow{\delta} \mathcal{T}_g (x)$ as $n \to \infty$. Hence, the theorem follows. \qed

**Theorem 4.7.** If $x_n \xrightarrow{\Delta} x$ as $n \to \infty$ in $\mathcal{B}_Y$, then $\mathcal{T}_g x_n \xrightarrow{\Delta} \mathcal{T}_g x$ as $n \to \infty$ in $\mathcal{B}_Y$.

**Proof.** Let $x_n \xrightarrow{\Delta} x$ as $n \to \infty$ in $\mathcal{B}_Y$. Then, by definition, we can find $\phi_n \in C^\infty(T)$ and $(\delta_n) \in \Delta$ such that $(x_n - x) \ast \delta_n = [\phi_n \ast \delta_k/\delta_k]$ and

$$\phi_n \to 0 \quad \text{as} \quad n \to 0 \quad \text{in} \quad C^\infty(T). \quad (4.13)$$

Applying the classical wavelet transform and using Lemma 4.3, we get

$$T_g \phi_n \to 0 \quad \text{as} \quad n \to 0 \quad \text{in} \quad \mathcal{F}(\mathbb{C}). \quad (4.14)$$

Hence, we get $\mathcal{T}_g x_n \xrightarrow{\Delta} \mathcal{T}_g x$ as $n \to \infty$ in $\mathcal{B}_Y$. \qed

**Lemma 4.8.** If $\eta \in \mathcal{F}(\mathbb{C})$ and $\phi \in C^\infty(T)$, then $R_g (\eta \ast \phi) = R_g \eta \ast \phi$.

**Proof.** Using Fubini’s theorem, we get

$$R_g (\eta \ast \phi) (x) = \int_0^{2\pi} \int_0^\infty g_a (x-b)(\eta \ast \phi)(b,a) \frac{dadb}{a}$$

$$= \int_0^{2\pi} \int_0^\infty g_a (x-b) \frac{dadb}{a} \int_0^{2\pi} \eta(b-t,a) \phi(t) dt$$

$$= \int_0^{2\pi} \phi(t) dt \int_0^{2\pi} \int_0^\infty g_a (x-b) \eta(b-t,a) \frac{dadb}{a}$$

$$= \int_0^{2\pi} \phi(t) dt \int_0^{2\pi} \int_0^\infty g_a ((x-t) - c) \eta(c,a) \frac{dadc}{a} \quad (b-t = c)$$

$$= \int_0^{2\pi} R_g \eta (x-t) \phi(t) dt$$

$$= (R_g \eta \ast \phi) (x). \quad (4.15)$$

\qed
Therefore, we can give the following definition.

**Definition 4.9.** Define \( R_g : \mathcal{B}_Y \rightarrow \mathcal{B}_T \) by

\[
R_g \left( \begin{bmatrix} \eta_n \\ \delta_n \end{bmatrix} \right) = \begin{bmatrix} R_g \eta_n \\ \delta_n \end{bmatrix}.
\] (4.16)

**Theorem 4.10.** The map \( R_g : \mathcal{B}_Y \rightarrow \mathcal{B}_T \) is linear.

**Theorem 4.11.** The map \( R_g : \mathcal{B}_Y \rightarrow \mathcal{B}_T \) is continuous with respect to \( \delta \)-convergence as well as \( \Delta \)-convergence.

Using Lemma 4.8 and Theorem 2.4, we get a proof similar to that of Theorems 4.6 and 4.7.

**Theorem 4.12** (an inversion formula). If \( x = \left[ \phi_n / \delta_n \right] \in \mathcal{B}_T \) such that \( \phi_n \in C_{\infty}^+ (\mathbb{T}) \) for all \( n \in \mathbb{N} \), then

\[
R_g \circ \mathcal{T}_g (x) = C_g^+ (\cdot) x.
\] (4.17)

**Proof.** Now,

\[
R_g \circ \mathcal{T}_g (x) = R_g \left( \begin{bmatrix} T_g \phi_n \\ \delta_n \end{bmatrix} \right) = \begin{bmatrix} (R_g \circ T_g) \phi_n \\ \delta_n \end{bmatrix}
\]

\[
= \begin{bmatrix} C_g^+ (\cdot) \phi_n \\ \delta_n \end{bmatrix} = C_g^+ (\cdot) \begin{bmatrix} \phi_n \\ \delta_n \end{bmatrix} = C_g^+ (\cdot) x.
\] (4.18)

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**References**


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