Let $B$ be a Galois algebra with Galois group $G$, $J_g = \{ b \in B \mid bx = g(x)b \text{ for all } x \in B \}$ for each $g \in G$, and $BJ_g = Be_g$ for a central idempotent $e_g$, $B_a$ the Boolean algebra generated by $\{0, e_g \mid g \in G\}$, $e$ a nonzero element in $B_a$, and $He = \{ g \in G \mid ee_g = e \}$. Then, a monomial $e$ is characterized, and the Galois extension $Be$, generated by $e$ with Galois group $He$, is investigated.

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1. Introduction. The Boolean algebra of central idempotents in a commutative Galois algebra plays an important role for the commutative Galois theory (see [1, 3, 6]). Let $B$ be a Galois algebra with Galois group $G$, $C$ the center of $B$, and $J_g = \{ b \in B \mid bx = g(x)b \text{ for all } x \in B \}$ for each $g \in G$. In [2], it was shown that $BJ_g = Be_g$ for some idempotent $e_g$ of $C$. Let $Ba$ be the Boolean algebra generated by $\{0, e_g \mid g \in G\}$. Then in [5], by using $Ba$, the following structure theorem for $B$ was given. There exist $\{e_i \in B_a \mid i = 1, 2, \ldots, m\}$ and some subgroups $H_i$ of $G$ such that $B = \oplus \sum_{i=1}^m Be_i \oplus Bf$ where $f = 1 - \sum_{i=1}^m e_i$, $Be_i$ is a central Galois algebra with Galois group $H_i$ for each $i = 1, 2, \ldots, m$, and $Bf = Cf$ which is a Galois algebra with Galois group induced by and isomorphic with $G$ in case $1 \neq \sum_{i=1}^m e_i$. In [4], let $K$ be a subgroup of $G$. Then, $K$ is called a nonzero subgroup of $G$ if $\prod_{k \in K} e_k \neq 0$ in $B_a$, and $K$ is called a maximal nonzero subgroup of $G$ if $K \subset K'$, where $K'$ is a nonzero subgroup of $G$ such that $\prod_{k \in K} e_k = \prod_{k \in K'} e_k$, then $K = K'$. We note that any nonzero subgroup is contained in a unique maximal nonzero subgroup of $G$. In [4], it was shown that there exists a one-to-one correspondence between the set of nonzero monomials in $B_a$ and the set of maximal nonzero subgroups of $G$. For a nonzero monomial $e$ in $B_a$ such that $He \neq \{1\}$, $Be$ is a central Galois algebra with Galois group $He$, if and only if $e$ is a minimal nonzero monomial in $B_a$. The purpose of the present paper is to characterize a monomial $e$ in $B_a$ in terms of the maximal nonzero subgroups of $G$. Then, the Galois extension $Be$, generated by a nonzero idempotent $e$ and by a monomial $e$ with Galois group $He$, is investigated, respectively. Let $G(e) = \{ g \in G \mid g(e) = e \}$ for each $e \neq 0$ in $B_a$. We will show that (1) $He$ is a normal subgroup of $G(e)$, and (2) $Be$ is a Galois extension of $(Be)^{He}$ with Galois group $He$ and $(Be)^{He}$ is a Galois extension of $(Be)^{G(e)}$ with Galois group $G(e)/He$. In particular, when $e$ is a monomial, $G(e) = N(He)$ (the normalizer
of $H_e$), and when $e$ is an atom (a minimal nonzero element) of $B_a$, $Be$ is a central Galois algebra over $Ce$ with Galois group $H_e$ and $Ce$ is a commutative Galois algebra with Galois group $G(e)/H_e$. This generalizes and improves the result of the components of $B$ in [5, Theorem 3.8] for a Galois algebra.

2. Definitions and notations. Let $B$ be a ring with 1, $C$ the center of $B$, $G$ an automorphism group of $B$ of order $n$ for some integer $n$, and $B^G$ the set of elements in $B$, fixed under each element in $G$. $B$ is called a Galois extension of $B^G$ with Galois group $G$ if there exist elements $\{a_i, b_i \in B, i = 1, 2, \ldots, m\}$ for some integer $m$ such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. $B$ is called a Galois algebra over $R$ if $B$ is a Galois extension of $R$ which is contained in $C$, and $B$ is called a central Galois extension if $B$ is a Galois extension of $C$. In this paper, we assume that $B$ is a Galois algebra with Galois group $G$. Let $f_g = \{b \in B \mid b x = g(x) b \text{ for all } x \in B\}$. In [2], it was shown that $Bf_g = B e_g$ for some central idempotent $e_g$ of $B$. We denote $(B_a; +, \cdot)$, the Boolean algebra generated by $\{0, e_g \mid g \in G\}$, where $e \cdot e' = e e'$ and $e + e' = e + e' - e e'$ for any $e$ and $e'$ in $B_a$. An order relation $\leq$ is defined as usual, that is, $e \leq e'$ in $B_a$ if $e \cdot e' = e$. Throughout, $e + e'$, for $e, e' \in B_a$, means the sum in the Boolean algebra $(B_a; +, \cdot), H_e = \{g \in G \mid e \leq e_g\}$ for an $e \neq 0$ in $B_a$, and a monomial $e$ in $B_a$ is $\prod_{g \in S} e_g \neq 0$ for some $S \subset G$.

3. The Boolean algebra. In this section, we will characterize a monomial $e$ in $B_a$ in terms of the maximal nonzero subgroups of $G$. We begin with several lemmas.

**Lemma 3.1.** Let $\{e_i, f \mid i = 1, 2, \ldots, m\}$ be given in [5, Theorem 3.8]. Then,

1. $\{e_i, f \mid i = 1, 2, \ldots, m\}$ is the set of all minimal elements of $B_a$ in case $f \neq 0$,
2. for each $e \neq 0$ in $B_a$, there exists a unique subset $Z_e$ of the set $\{1, 2, \ldots, m\}$ such that $e = \sum_{i \in Z_e} e_i$ or $e = \sum_{i \in Z_e} e_i + f$.

**Proof.** (1) By the proof of [5, Theorem 3.8], either $e_i = \prod_{g \in H_t} e_g$, where $H_t$ is a maximum subset (subgroup) of $G$ such that $\prod_{g \in H_t} e_g \neq 0$, or $e_i = (1 - \sum_{j=1}^t e_j) \prod_{g \in H_t} e_g$ for some $t < i$, where $H_t$ is a maximum subset (subgroup) of $G$ such that $(1 - \sum_{j=1}^t e_j) \prod_{g \in H_t} e_g \neq 0$; so, either $e_i$ is a minimal element of $B_a$ or $e_i$ is a minimal element of $(1 - \sum_{j=1}^t e_j) B_a$. Noting that any minimal element in $(1 - \sum_{j=1}^t e_j) B_a$ is also a minimal element in $B_a$, we conclude that each $e_i$ is a minimal element in $B_a$. Next, we show that $f$ is also a minimal element of $B_a$ in case $f \neq 0$. In fact, by the proof of [5, Theorem 3.8], $e_g f = 0$ for any $g \neq 1$ in $G$; so, for any $e \in B_a$, $e f = 0$ or $e f = f$. This implies that $f$ is a minimal element of $B_a$ in case $f \neq 0$. Moreover, $\sum_{i=1}^m e_i + f = 1$; so, $\{e_i, f \mid i = 1, 2, \ldots, m\}$ is the set of all minimal elements of $B_a$ in case $f \neq 0$.

(2) Since $1 = \sum_{i=1}^m e_i + f$, a sum of all minimal elements of $B_a$, the statement is immediate.

\[\Box\]
LEMMA 3.2. Let $e$ be a nonzero element in $B_a$. Then,
(1) there exists a monomial $e'$ of $B_a$ such that $e \leq e'$ and $H_e = H_{e'}$,
(2) $H_e$ is a maximal nonzero subgroup of $G$.

PROOF. (1) For any nonzero element $e$ in $B_a$, let $e' = \prod_{g \in H_e} e_g$. We claim that $e \leq e'$ and $H_e = H_{e'}$. In fact, for any $h \in H_e$, $e \leq e_h$; so, $e \leq \prod_{h \in H_e} e_h = e'$. Moreover, for any $h \in H_e$, $e_h \geq \prod_{g \in H_e} e_g = e'$; so, $h \in H_{e'}$. Hence, $H_e \subseteq H_{e'}$. On the other hand, for any $h \in H_{e'}$, $e_h \geq \prod_{g \in H_e} e_g \geq e$; so, $h \in H_e$. Thus, $H_{e'} \subseteq H_e$. Therefore, $H_e = H_{e'}$.

(2) By [4, Theorem 3.2], $H_{e'}$ is a maximal nonzero subgroup of $G$ for $e'$ is a monomial. Hence, $H_e (= H_{e'})$ is a maximal nonzero subgroup of $G$. \hfill \Box

Next is an expression of $H_e$ for a nonzero $e \in B_a$.

THEOREM 3.3. For any $e \neq 0$ in $B_a$, $H_e = \cap_{i \in Z_e} H_{e_i}$ or $H_1$, where $e = \sum_{i \in Z_e} e_i$ or $e = \sum_{i \in Z_e} e_i + f$ as given in Lemma 3.1(2).

PROOF. We first show that for $e = e' + e''$ for some $e', e'' \neq 0$ in $B_a$, $H_e = H_{e'} \cap H_{e''}$. In fact, since $e \geq e'$ and $e \geq e''$, we have $H_e \subseteq H_{e'} \cap H_{e''}$. Conversely, for any $g \in H_{e'} \cap H_{e''}$, $e_g \geq e'$ and $e_g \geq e''$; so, $e_g \geq e' + e'' = e$. Hence, $g \in H_e$; so, $H_e = H_{e'} \cap H_{e''}$. Therefore, by induction, if $e = \sum_{i \in Z_e} e_i$, then $H_e = \cap_{i \in Z_e} H_{e_i}$. Now, by Lemma 3.1, for any $e \neq 0$ in $B_a$, $e = \sum_{i \in Z_e} e_i$ or $e = \sum_{i \in Z_e} e_i + f$. Similarly, if $e = \sum_{i \in Z_e} e_i + f$, then $H_e = H(\sum_{i \in Z_e} e_i + f) = (\cap_{i \in Z_e} H_{e_i}) \cap H_f$. But, for $g \in G$ such that $e_g \neq 1$, $e_g f = 0$; so, $H_f = H_1$. Therefore, $H_e = (\cap_{i \in Z_e} H_{e_i}) \cap H_1 = H_1$ for $H_1 \subseteq H_{e_i}$ for each $i$. \hfill \Box

We observe that there exist some $e \neq 0$ such that $H_e = \cap_{i \in Z_e} H_{e_i}$ and $H_e \subset H_{e_j}$ for some $j \notin Z_e$, and that not all $e \neq 0$ are monomials. Next, we identify which element $e \neq 0$ in $B_a$ is a monomial. Two characterizations are given. We begin with a definition.

DEFINITION 3.4. An $e \neq 0$ in $B_a$ is called a maximal $G$-element if $H_e \neq H_1$ and, for any $e' \in B_a$ such that $e \leq e'$ and $H_e = H_{e'}$, $e = e'$.

LEMMA 3.5. (1) If $e \neq 0$ such that $e f = 0$, then $e = \sum_{i \in Z_e} e_i$.
(2) If $e$ is a monomial, $e = \prod_{g \in S} e_g$ for some $S \subset G$, then $e = 1$ or $e = \sum_{i \in Z_e} e_i$.

PROOF. (1) By Lemma 3.1, $e = \sum_{i \in Z_e} e_i$ or $e = \sum_{i \in Z_e} e_i + f$. If $e \neq \sum_{i \in Z_e} e_i$, then $e = \sum_{i \in Z_e} e_i + f$ and $f \neq 0$. But then, $f = (\sum_{i \in Z_e} e_i + f) - e f = 0$. This is a contradiction. Hence, $e = \sum_{i \in Z_e} e_i$.
(2) In case $e = 1$, we are done. In case $e \neq 1$. Since $e_g f = 0$ for each $g \in G$ such that $e_g \neq 1$, $e f = \prod_{g \in S} e_g f = 0$. Thus, by (1), $e = \sum_{i \in Z_e} e_i$. \hfill \Box

THEOREM 3.6. Keeping the notations of Lemma 3.1 for any $e \neq 0, 1$ in $B_a$, the following statements are equivalent:
(1) $e = \prod_{g \in S} e_g$ for some $S \subset G$, a monomial in $B_a$;
(2) $e$ is a maximal $G$-element in $B_a$;
(3) \( e = \sum_{i \in Z_e} e_i \) where \( \{ e_i \mid i \in Z_e \} \) are all atoms such that \( H_e \subset H_{e_i} \) and \( H_e \neq H_1 \).

**Proof.** (1)⇒(2). Since \( e \) is a monomial and \( e \neq 1 \), \( e = \prod_{g \in H_e} e_g \) where \( e_g \neq 1 \) for some \( g \in H_e \). Thus, \( H_e \neq H_1 \). Next, for any \( e' \) such that \( e \leq e' \) and \( H_e = H_{e'} \),

\[
eq e' \leq \prod_{g \in H_e} e_g = \prod_{g \in H_e} e_g = e. \tag{3.1}
\]

Hence, \( e = e' \). This implies that \( e \) is a maximal \( G \)-element in \( B_\alpha \).

(2)⇒(1). Let \( e \) be a maximal \( G \)-element and \( e' = \prod_{g \in H_e} e_g \). Then, by Lemma 3.2, \( e \leq e' \) and \( H_e = H_{e'} \). But \( e \) is a maximal \( G \)-element; so, \( e = e' \) which is a monomial.

(1)⇒(3). By Lemma 3.5, \( e = \sum_{i \in Z_e} e_i \). Now, let \( e_j \) be an atom such that \( H_e \subset H_{e_j} \). Then, \( e_j \leq \prod_{g \in H_{e_j}} e_g \leq \prod_{g \in H_e} e_g \). But, by hypothesis, \( e \) is a monomial; so, \( e = \prod_{g \in H_e} e_g \). Hence, \( e_j \leq e \). This implies that \( e_j \) is a term in \( e \). Thus, \( e = \sum_{i \in Z_e} e_i \) and \( \{ e_i \mid i \in Z_e \} \) are all atoms such that \( H_e \subset H_{e_i} \). Moreover, since \( e = \prod_{g \in S} e_g \neq 1 \), there exists \( g \in G \) such that \( e \leq e_g \neq 1 \). Thus, \( g \in H_e \) and \( g \notin H_1 \). Therefore, \( H_e \neq H_1 \).

(3)⇒(1). Let \( e' = \prod_{g \in H_e} e_g \). Then, by Lemma 3.2, \( e \leq e' \) and \( H_e = H_{e'} \). Since \( H_e \neq H_1 \), \( H_{e'} \neq H_1 \). Also, since \( e' \) is a monomial, \( e' = \sum_{j \in Z_{e'}} e_j \) by Lemma 3.5(2). Now, suppose that \( e \neq e' \). Then, there is a \( j \in Z_{e'} \) but \( j \notin Z_e \), that is, \( e_j \) is a term of \( e' = \sum_{j \in Z_{e'}} e_j \) but not a term of \( e = \sum_{i \in Z_e} e_i \). But then, \( H_e = H_{e'} = \cap_{j \in Z_{e'}} H_{e_j} \subset H_{e_j} \) such that \( j \notin Z_e \). This contradicts the hypothesis that \( e = \sum_{i \in Z_e} e_i \) where \( \{ e_i \mid i \in Z_e \} \) are all atoms such that \( H_e \subset H_{e_i} \). Thus, \( e = e' \) which is a monomial in \( B_\alpha \). 

4. Galois extensions. In [5], it was shown that \( B_e \) is a central Galois algebra with Galois group \( H_e \) for any atom \( e \neq f \) of \( B_\alpha \). Also, for any \( e \neq 0 \) in \( B_\alpha \), \( B_e \) is a Galois extension of \( (B_e)^e \) with Galois group \( G(e) \) where \( G(e) = \{ g \in G \mid g(e) = e \} \) (see [5, Lemma 3.7]). In this section, we are going to show that, for any \( e \neq 0 \) in \( B_\alpha \) (not necessary an atom), (1) \( H_e \) is a normal subgroup of \( G(e) \), and (2) \( B_e \) is a Galois extension of \( (B_e)^H_e \) with Galois group \( H_e \) and \( (B_e)^H_e \) is a Galois extension of \( (B_e)^G(e) \) with Galois group \( G(e)/H_e \). This generalizes and improves the result for \( B_e \) when \( e \) is an atom of \( B_\alpha \) as given in [5, Theorem 3.8]. In particular, for a monomial \( e \), \( G(e) = N(H_e) \), the normalizer of \( H_e \) in \( G \).

**Lemma 4.1.** Let \( e \neq 0 \) in \( B_\alpha \). Then, \( H_e \) is a normal subgroup of \( G(e) \) where \( G(e) = \{ g \in G \mid g(e) = e \} \).

**Proof.** We first claim that \( H_e \subset G(e) \). In fact, by Lemma 3.1, for any \( e \neq 0 \) in \( B_\alpha \), there exists a unique subset \( Z_e \) of the set \( \{ 1, 2, \ldots, m \} \) such that \( e = \sum_{i \in Z_e} e_i \) or \( e = \sum_{i \in Z_e} e_i + f \) where \( e_i \) are given in Lemma 3.1. Moreover, for each \( i \),
\[ e_i = \prod_{h \in H_{e_i}} e_h \text{ or } e_i = (1 - \sum_{j=1}^{t} e_j) \prod_{g \in H_{e_i}} e_g \text{ for some } t < i. \]
Noting that \( g \) permutes the set \( \{e_i \mid i = 1, 2, \ldots, t\} \) for each \( g \in G \) by the proof of [5, Theorem 3.8], we have, for each \( g \in G \),

\[ g(e_i) = g\left( \prod_{h \in H_{e_i}} e_h \right) = \prod_{h \in H_{e_i}} e_{ghg^{-1}} \geq \prod_{h \in H_{e_i}} e_g e_h e_{g^{-1}} = e_g e_i e_{g^{-1}} \]  

(4.1)

or

\[ g(e_i) = g\left( \left(1 - \sum_{j=1}^{t} e_j\right) \prod_{h \in H_{e_i}} e_h \right) = \left(1 - \sum_{j=1}^{t} e_j\right) \prod_{h \in H_{e_i}} e_{ghg^{-1}} \]

\[ \geq \left(1 - \sum_{j=1}^{t} e_j\right) \prod_{h \in H_{e_i}} e_g e_h e_{g^{-1}} \]  

(4.2)

= \left(1 - \sum_{j=1}^{t} e_j\right) \prod_{h \in H_{e_i}} e_{h} e_{g^{-1}} = e_g e_i e_{g^{-1}}.

Now, in case \( e = \sum_{i \in Z_e} e_i \), for any \( h \in H_e \),

\[ e = e_h e_i e_{h^{-1}} = \sum_{i \in Z_e} e_h e_i e_{h^{-1}} \leq \sum_{i \in Z_e} h(e_i) = h(e). \]  

(4.3)

Thus, \( h(e) = e \) using Lemma 3.1(2). Noting that \( g \) permutes the set \( \{e_i \mid i = 1, 2, \ldots, m\} \) for each \( g \in G \), we have \( g(f) = f \) for each \( g \in G \). Thus, we have \( h(e) = e \) for each \( h \in H_e \) in case \( e = \sum_{i \in Z_e} e_i + f \). This proves that \( H_e \subset G(e) \).

Next, we show that \( H_e \) is a normal subgroup of \( G(e) \). Since for each \( g \in G \), \( g(e_i) \) is also an atom, \( g(e) = e \) (i.e., \( g \in G(e) \)) implies that \( g \) permutes the set \( \{e_i \mid i \in Z_e\} \). Therefore, for each \( i \in Z_e \), \( g(e_i) = e_j \) and \( gH_{e_i} g^{-1} = H_{e_j} \) for some \( j \in Z_e \). But, by Theorem 3.3, \( H_e = \cap_{i \in Z_e} H_{e_i} \) (or \( H_e = H_1 \) which is normal); so, for any \( g \in G(e) \), \( gH_{e_i} g^{-1} = g(\cap_{i \in Z_e} H_{e_i}) g^{-1} = \cap_{i \in Z_e} gH_{e_i} g^{-1} = \cap_{j \in Z_e} H_{e_j} = H_e \).

Therefore, \( H_e \) is a normal subgroup of \( G(e) \).

\[ \Box \]

**Theorem 4.2.** Let \( e \) be a nonzero element in \( B_a \). Then,

1. \( Be \) is a Galois extension of \((Be)^{G(e)}\) with Galois group \( G(e) \),

2. \( Be \) is a Galois extension of \((Be)^{H_e}\) with Galois group \( H_e \) and \((Be)^{H_e}\) is a Galois extension of \((Be)^{G(e)}\) with Galois group \( G(e)/H_e \).

**Proof.** (1) Since \( B \) is a Galois algebra with Galois group \( G \), \( B \) is a Galois extension with Galois group \( G(e) \). But \( g(e) = e \) for each \( g \in G(e) \); so, by [5, Lemma 3.7], \( Be \) is a Galois extension of \((Be)^{G(e)}\) with Galois group \( G(e) \).

(2) Clearly, \( Be \) is a Galois extension of \((Be)^{H_e}\) with Galois group \( H_e \) by part (1). Next, we claim that \( |H_{e_1}| \), the order of \( H_{e_1} \), is a unit in \( Be \). In fact, by [5, Theorem 3.8], for each atom \( e_1 \) of \( B_a \), \( Be_1 \) is a central Galois algebra over \( Ce_1 \) with Galois group \( H_{e_1} \); so, \( |H_{e_1}| \), the order of \( H_{e_1} \), is a unit in \( Be_1 \) (see [2, Corollary 3]). Hence, \( |H_{e_1}| \) is a unit in \( Be \) if \( e = \sum_{i \in Z_e} e_i \). If \( e = \sum_{i \in Z_e} e_i + f \) and \( f \neq 0 \), then \( H_e = H_1 = \{g \in G \mid e_g = 1\} = \{g \in G \mid g(c) = c \text{ for each } c \in C\} \). Hence, by
Lemma 4.1 shows that, for any nonzero element \( e \) in \( B_\alpha \), \( G(e) \) is contained in (not necessarily equal to) the normalizer \( N(H_e) \) of \( H_e \) in \( G \). Next, we want to show that \( G(e) = N(H_e) \) when \( e \) is a monomial. Consequently, for any nonzero element \( e \) in \( B_\alpha \), \( Be \) is embedded in a Galois extension \( Be' \) with the same Galois group \( H_e \), and \( (Be')^{H_e} \) is a Galois extension of \( (Be')^{G(e')} \) with Galois group \( G(e')/H_e \) such that \( G(e') = N(H_e) \) for some monomial \( e' \) in \( B_\alpha \).

**Lemma 4.3.** Let \( e \) be a nonzero element in \( B_\alpha \). Then, there exists a monomial \( e' \) in \( B_\alpha \) such that \( e \leq e' \), \( H_e = H_e' \), and \( N(H_e) = G(e') \) where \( G(e') = \{ g \in G \mid g(e') = e' \} \) and \( N(H_e) \) is the normalizer of \( H_e \) in \( G \).

**Proof.** By Lemma 3.2, there exists a monomial \( e' \) in \( B_\alpha \) such that \( e \leq e' \) and \( H_e = H_e' \); so, it suffices to show that \( N(H_e) = G(e') \). For any \( g \in N(H_e) \), \( g \in N(H_e') \); so, by Theorem 3.3, \( H_e' = gH_e'g^{-1} = g(\cap_{i \in Z_e} H_{ei}^i)g^{-1} = \cap_{i \in Z_e} gH_{ei}^ig^{-1} = gH_{ei}g^{-1} = g \sum_{i \in Z_e} gH_{ei}g^{-1} = H_{g(e_i)} = H_{g(e')} \). Noting that \( e' \) is a monomial, we have \( g(e') = e' \) by Lemma 3.2, that is, \( g \in G(e') \). This implies that \( N(H_e) \subset G(e') \). Conversely, \( G(e') \subset N(H_e) \) by Lemma 4.1. But \( H_e = H_e' \); so, \( G(e') \subset N(H_e') \) is \( N(H_e) \). Therefore, \( N(H_e) = G(e') \).

**Theorem 4.4.** Let \( e \) be a nonzero element in \( B_\alpha \). Then, there exists a monomial \( e' \) in \( B_\alpha \) such that \( Be \) is embedded in \( Be' \), \( Be' \) is a Galois extension of \( (Be')^{H_e} \) with Galois group \( H_e \), and \( (Be')^{H_e} \) is a Galois extension of \( (Be')^{N(H_e)} \) with Galois group \( N(H_e)/H_e \).

**Proof.** By Lemma 4.3, there exists a monomial \( e' \) in \( B_\alpha \) such that \( e \leq e' \), \( H_e \) is a normal subgroup of \( G(e') \), and \( N(H_e) = G(e') \). Hence, \( Be \subset Be' \). But \( Be' \) is a Galois extension of \( (Be')^{H_e} \) with Galois group \( H_e \) and \( (Be')^{H_e} \) is a Galois extension of \( (Be')^{G(e')} \) with Galois group \( G(e')/H_e \) by Theorem 4.2; so, Theorem 4.4 holds.

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