EQUIVALENCE CLASSES OF THE 3RD GRASSMAN SPACE
OVER A 5-DIMENSIONAL VECTOR SPACE

KULDIP SINGH
Department of Mathematics
The University of New Brunswick
Fredericton, N.B., Canada E3B 5A3

(Received June 10, 1977)

ABSTRACT. An equivalence relation is defined on $\Lambda^r V$, the $r$th Grassman space
over $V$ and the problem of the determination of the equivalence classes defined by this relation is considered. For any $r$ and $V$, the decomposable elements form an equivalence class. For $r = 2$, the length of the element determines the equivalence class that it is in. Elements of the same length are equivalent, those of unequal lengths are inequivalent. When $r \geq 3$, the length is no longer a sufficient indicator, except when the length is one. Besides these general questions, the equivalence classes of $\Lambda^3 V$, when $\dim V = 5$ are determined.

KEY WORDS AND PHRASES. Grassman space, equivalent classes, representation of equivalent classes.

Suppose $V$ is a finite dimensional vector space over an arbitrary field $F$ and $r$ is a positive integer. Consider $\Lambda^r V$, the $r$th Grassman space over $V$. We define an equivalence relation on $\Lambda^r V$ as follows: If $X$ and $Y$ are in $\Lambda^r V$, we write $X \sim Y$ iff $\exists$ a non-singular linear transformation $T: V \rightarrow V$ such that $C_r(T)X = Y$, where $C_r(T)$ is the $r$th exterior product of $T$. Using the facts that if $T$ and $S$ are two linear transformations of $V$, then $C_r(T)C_r(S) = C_r(TS)$ and if $T$ is non-singular, then $C_r(T^{-1}) = C_r(T)^{-1}$, it follows that the above relation is an equivalence relation.

We consider the problem of determining the number of equivalence classes, into which the set $\Lambda^r V$ is decomposed, along with a system of distinct representatives of these equivalence classes.

DEFINITIONS. 1. If $X \in \Lambda^r V$ and $X = x_1 \wedge \ldots \wedge x_r$, we say $X$ is decomposable.

2. If $X \in \Lambda^r V$, we define its length, to be denoted by $\ell(X)$ as $\ell(X) = \min\{m | X$ is a sum of $m$ decomposable elements of $\Lambda^r V\}$.

3. If $X \in \Lambda^r V$, we define a subspace $[X]$ of $V$ as $[X] = \cap\{U | U$ is a subspace of $V$ and $X \in \Lambda^r U\}$.

4. If $X \in \Lambda^r V$, we define the rank of $X$ to be denoted by $\rho(X)$ as $\rho(X) = \dim[X]$.

PROPOSITION 1. If $X, Y \in \Lambda^r V$ and $X \sim Y$, then (i) $\ell(X) = \ell(Y)$, (ii) $\rho(X) = \rho(Y)$.

PROOF. (i) Let $T: V \rightarrow V$ be a n.s.l.t. such that $C_r(T)X = Y$.
If $\ell(X) = s$ $X = \sum_{i=1}^{s} x_i$, where $x_i \in \Lambda^r V$ and $\ell(x_i) = 1$.

Then $Y = C_r(T)X = \sum_{i=1}^{s} C_r(T)x_i$. This implies $\ell(Y) \leq s = \ell(X)$. Similarly $Y \sim X$ implies $\ell(Y) \leq \ell(X)$ and this proves (i).
(ii) We first remark that if U and W are subspaces of V, then $X \in \Lambda^r U$ implies $Y \in \Lambda^r T(U)$ and $Y \in \Lambda^r W$ implies $X \in \Lambda^r T^{-1}(W)$, where $T : V \to V$ is a n.s.l.t. such that $Y = C_r(T)X$. From this remark, it follows easily that $[Y] = T[X]$ and hence $P(X) = P(Y)$.

**Proposition 2.** If U and W are subspaces of V, then $\Lambda^r U \cap \Lambda^r W = \Lambda^r (U \cap W)$.

**Proof.** Clearly $\Lambda^r (U \cap W) \subseteq (\Lambda^r U) \cap (\Lambda^r W)$. To prove the inclusion in the other direction, let $x_1, x_2, \ldots, x_k$ be a basis of $U \cap W$ and extend it to a basis $x_1, \ldots, x_k, y_1, \ldots, y_s$ of U and a basis $x_1, \ldots, x_k, z_1, \ldots, z_t$ of W. Then $x_1, \ldots, x_k, y_1, \ldots, y_s, z_1, \ldots, z_t$ is a basis of $U + W$.

If $A = \{x_i \wedge y_j | 1 \leq i < j \leq k\}$, $B = \{y_i \wedge y_j | 1 \leq i < j \leq s\}$, $C = \{z_i \wedge z_j | 1 \leq i < j \leq t\}$, $D = \{x_i \wedge y_j | 1 \leq i < k; 1 \leq j \leq s\}$, $E = \{x_i \wedge z_j | 1 \leq i < k; 1 \leq j \leq t\}$, $F = \{y_i \wedge z_j | 1 \leq i < s; 1 \leq j \leq t\}$, then the sets $A, A \cup B \cup D, A \cup C \cup E, A \cup B \cup C \cup D \cup E \cup F$ form bases of $\Lambda^r (U \cap W), \Lambda^r U, \Lambda^r W$ and $\Lambda^r (U + W)$ respectively. If $X \in (\Lambda^r U) \cap (\Lambda^r W)$, then

$$X = \sum_{A} a_{ij} x_i \wedge x_j + \sum_{B} b_{ij} y_i \wedge y_j + \sum_{C} c_{ij} z_i \wedge z_j + \sum_{D} d_{ij} x_i \wedge y_j \text{ and also}$$

$$X = \sum_{A} a_{ij} x_i \wedge x_j + \sum_{C} c_{ij} z_i \wedge z_j + \sum_{D} d_{ij} x_i \wedge y_j \text{ and also}$$

$$X = \sum_{A} a_{ij} x_i \wedge x_j + \sum_{C} c_{ij} z_i \wedge z_j + \sum_{E} e_{ij} x_i \wedge z_j.$$ 

Hence $a_{ij} = a_{ij}$ and $b_{ij} = d_{ij} = c_{ij} = e_{ij} = 0$ for all the appropriate values of the indices $i$ and $j$. Thus $X \in \Lambda^r (U \cap W)$.

**Remark 1.** The result of Proposition 2 holds for any number of subspaces of V.

**Remark 2.** If $X \in \Lambda^r V$ and $E = \{U | U \text{ is a subspace of } V, X \in \Lambda^r U\}$, then $\Lambda^r [X] = \Lambda^r (\sqcup U_{E}) = \sqcup (\Lambda^r U)$. Thus $X \in \Lambda^r [X]$ and $[X]$ is the smallest such subspace of V.

**Proposition 3.** Let $X \in \Lambda^2 V$, $\ell(X) = k$ and $X = \sum_{i=1}^{k} x_i \wedge y_i$, then $x_1, \ldots, x_k, y_1, \ldots, y_k$ are linearly independent.

**Proof.** If not, then one of them (say) $y_k$ is a linear combination of the
remaining $x_1, \ldots, x_k, y_1, \ldots, y_{k-1}$. Let $y_k = \sum_{i=1}^{k-1} a_i x_i + \sum_{j=1}^{k-1} b_j y_j$. Then $x_k \wedge y_k = \sum_{i=1}^{k-1} a_i x_k \wedge x_i + \sum_{j=1}^{k-1} b_j x_k \wedge y_j$. Hence $X$ can be written as
\[ X \cong \sum_{i=1}^{k-1} (a_i x_i + b_j y_j). \]

Hence $\ell(X) \leq k-1$, a contradiction.

**Remark 3.** If $X \in \Lambda^2 V$, $\ell(X) = k$ and $X = \sum_{i=1}^{k} x_i \wedge y_i$, then $[X] = <x_1, \ldots, x_k, y_1, \ldots, y_k>$.

**Proof.** Let $U = <x_1, \ldots, x_k, y_1, \ldots, y_k>$; then $[X] \subseteq U$. By Proposition 3, $\dim U = 2k$. Also $X \in \Lambda^2[X]$; let $X = \sum_{i=1}^{k} x_i \wedge y_i$. Since $X \in \Lambda^2 V$, $\ell(X) = k$ and $X = \sum_{i=1}^{k} x_i \wedge y_i$, then by Remark 3, $[X] = <x_1, \ldots, x_k, y_1, \ldots, y_k>$.

**Proposition 4.** If $X, Y \in \Lambda^2 V$, $P(X) = P(Y)$, then $X \sim Y$.

**Proof.** Let $X = \sum_{i=1}^{k} x_i \wedge y_i$ and $Y = \sum_{j=1}^{s} x_j \wedge y_j$. Then $X \sim Y$.

**Proposition 5.** If $X \in \Lambda^r V$, $\ell(X) = 2$, $X = x_1 \wedge \ldots \wedge x_r \wedge y_1 \wedge \ldots \wedge y_r$, then $X = <x_1, \ldots, x_r, y_1, \ldots, y_r>$.

**Proof.** Let $U = <x_1, \ldots, x_r, y_1, \ldots, y_r>$; then $[X] \subseteq U$. If $[X] \neq U$, then at least one element (say) $x_1$ is not in $[X]$. Let $B$ be a basis of $[X]$ and extend $\{x\} \cup B$ to a basis of $U$. Let $W$ be a complement of $<x_1>$ in $U$, containing $[X]$, i.e., $U = <x_1> \oplus W$, $[X] \subseteq W$. Let $x_2 = a_2 x_1 + w_1$, $2 \leq i \leq r$ and $y_j = b_j x_1 + w'_j$, $1 \leq j \leq r$, where $w_i, w_j \in W$. Then $X = X_1 + X_2$, where $X_1 \in x_1 \wedge (\Lambda^{r-1} W)$ and $X_2 \in \Lambda^r W$, and $\ell(X_1) = 1$, $i = 1, 2$. But $U = <x_1> \oplus W = \Lambda^r U = x_1 \wedge (\Lambda^{r-1} W) \oplus \Lambda^r W$. Also $X \in \Lambda^r [X] \subseteq \Lambda^r W$, hence
Thus $X_1 = 0$ and $X = X_2 \Rightarrow \ell(X) = 1$, a contradiction.

Hence $[X] = U = \langle x_1, \ldots, x_r, y_1, \ldots, y_r \rangle$.

**Note:** The above proposition is true also for $\ell(X) = k$.

**PROPOSITION 6.** If $X,Y \in \Lambda^r V$, $\ell(X) = \ell(Y) = 2$, $P(X) = P(Y)$, then $X \sim Y$.

**PROOF.** Let $X = x_1 \wedge \ldots \wedge x_r + y_1 \wedge \ldots \wedge y_r$, $U_1 = \langle x_1, \ldots, x_r \rangle$, $U_2 = \langle y_1, \ldots, y_r \rangle$, then by Proposition 4, $[X] = U_1 + U_2$. Let $z_1, \ldots, z_k$ be a basis of $U_1 \cap U_2$, and extend it to a basis $z_1, \ldots, z_k, u_1, \ldots, u_s$, where $k + s = r$ of $U_1$ and to a basis $z_1, \ldots, z_k, v_1, \ldots, v_s$ of $U_2$. Then

$P(X) = k + 2s$. Since $x_1, \ldots, x_r$ and $z_1, \ldots, z_k, u_1, \ldots, u_s$ are two bases of $U_1$, hence $x_1 \wedge \ldots \wedge x_r = az_1 \wedge \ldots \wedge z_k \wedge u_1 \wedge \ldots \wedge u_s = z_1 \wedge \ldots \wedge z_k \wedge u_1 \wedge \ldots \wedge u_s$, where $\overline{u}_1 = au_1$. Similarly $y_1 \wedge \ldots \wedge y_r = bz_1 \wedge \ldots \wedge z_k \wedge v_1 \wedge \ldots \wedge v_s = z_1 \wedge \ldots \wedge z_k \wedge \overline{v}_1 \wedge \ldots \wedge \overline{v}_s$, where $\overline{v}_1 = b v_1$. Hence $X = z_1 \wedge \ldots \wedge z_k \wedge (\overline{u}_1 \wedge u_2 \wedge \ldots \wedge u_s + \overline{v}_1 \wedge v_2 \wedge \ldots \wedge v_s)$, where $z_1, \ldots, z_k, \overline{u}_1, u_2, \ldots, u_s, \overline{v}_1, v_2, \ldots, v_s$ is a basis of $[X]$.

Similarly $Y = z_1' \wedge \ldots \wedge z_k' \wedge (\overline{u}_1' \wedge u_2' \wedge \ldots \wedge u_s' + \overline{v}_1' \wedge v_2' \wedge \ldots \wedge v_s')$, where $z_1', \ldots, z_k', \overline{u}_1', u_2', \ldots, u_s', \overline{v}_1', v_2', \ldots, v_s'$ is a basis of $[Y]$.

Define $T: V \rightarrow V$, a linear transformation

$Tz_i = z_i'$, $T\overline{u}_1 = \overline{u}_1'$, $Tu_1 = u_1'$, $T\overline{v}_1 = \overline{v}_1'$, $Tv_i = v_i'$, for $i = 2, 3, \ldots, s$.

Then $C_r(T)X = Y$; hence $X \sim Y$.

**REMARK 4.** Let $X \in \Lambda^r V$, $\ell(X) = 2$, then $r + 1 \leq \rho(X) \leq 2r$.

**PROOF.** If $X = x_1 \wedge \ldots \wedge x_r + y_1 \wedge \ldots \wedge y_r$, then $[X] = \langle x_1, \ldots, x_r, y_1, \ldots, y_r \rangle = U_1 + U_2$, where $U_1 = \langle x_1, \ldots, x_r \rangle$, $U_2 = \langle y_1, \ldots, y_r \rangle$. $U_1 \neq U_2$, for otherwise $y_1 \wedge \ldots \wedge y_r = ax_1 \wedge \ldots \wedge x_r$, where $a$ is a scalar and $\ell(X) = 1$.

$P(X) = 2r - \dim U_1 \cap U_2$. Hence $r + 1 \leq P(X) \leq 2r$.

**THEOREM 1.** Let $E(2, s) = \{X|X \in \Lambda^r V, \ell(X) = 2, P(X) = s\}$, then $E(2, s), s = r + 1, r + 2, \ldots, 2r$ are all the equivalence classes on the set of all vectors of $\Lambda^r V$, of length 2.

**PROOF.** Follows from Proposition 6 and Remark 4.
PROPOSITION 7. Let \( 0 \neq X \in \Lambda^r V \) and \( x \in V \) such that \( x \wedge X = 0 \); then \( x \in [X] \).

PROOF. Let \( x_1, x_2, \ldots, x_m \) be a basis of \([X]\). Then \( \{x_\alpha \mid \alpha \in Q_{r,m}\} \) is a basis of \( \Lambda^r[X] \), where \( Q_{r,m} \) is a set of all the strictly decreasing sequences of length \( r \) on the integers \( 1, 2, \ldots, m \). \( \det X \neq 0 \), then \( (X) \neq 0 \). If \( x \notin [X] \), then \( \{x \wedge x_\alpha \mid \alpha \in Q_{r,m}\} \) is a part of a basis of \( \Lambda^{r+1} \langle x, [X]\rangle \). Thus \( x \wedge X = 0 \) \( \Rightarrow \) \( a_\alpha = 0 \) \( \forall \alpha \in Q_{r,m} \Rightarrow X = 0 \), a contradiction.

PROPOSITION 8. If \( 0 \neq X \in \Lambda^r V \) and \( x \notin [X] \), then \( [x \wedge X] = \langle x \rangle \oplus [X] \).

PROOF. By Proposition 7, \( x \wedge X \neq 0 \). Again by Proposition 7, since \( x \wedge (x \wedge X) = 0 \), hence \( x \in [x \wedge X] \). Clearly \( [x \wedge X] \subseteq \langle x \rangle \oplus [X] \). Let \( x, x_1, \ldots, x_k \) be a basis of \([x \wedge X]\) and extend it to a basis \( x, x_1, \ldots, x_k, x_{k+1}, \ldots, x_m \) of \( \langle x \rangle \oplus [X] \). If \( U = \langle x_1, \ldots, x_k \rangle \), then \( [x \wedge X] = \langle x \rangle \oplus U \), \( U \subseteq [X] \).

\( \Lambda^{r+1} [x \wedge X] = x \wedge (\Lambda^r U) \oplus \Lambda^{r+1} U \). Let \( x \wedge X = x \wedge u + v \), where \( u \in \Lambda^r U \) and \( v \in \Lambda^{r+1} U \). Thus \( x \wedge v = 0 \). If \( v \neq 0 \), then by Proposition 7, \( x \in [v] \subseteq U \), a contradiction. Hence \( v = 0 \) and thus \( x \wedge X = x \wedge u \). Then \( x \wedge (X-u) = 0 \). If \( X-u \neq 0 \), then by Proposition 7, \( x \in [X-u] \). Now \( X \in \Lambda^r[X] \) and \( u \in \Lambda^r U \subseteq \Lambda^r [X] \), thus \( X - u \in \Lambda^r [X] \). Hence \( [X-u] \subseteq [X] \). Thus \( x \in [X-u] \Rightarrow x \in [X] \), which is a contradiction and therefore \( X-u = 0 \); i.e., \( X = u \in \Lambda^r U \). Hence \( [X] \subseteq U \).

Also \( U \subseteq [X] \), hence \( U = [X] \) and \( [x \wedge X] = \langle x \rangle \oplus [X] \).

PROPOSITION 9. Suppose \( X \in \Lambda^2 V \), \( \ell(X) = 2 \), \( x_1, x_2 \) are linearly independent vectors in \([X]\). Then \( \exists y_1, y_2 \in [X] \) and \( \lambda \in F \exists X \) has one and only one of the following representations: (i) \( X = x_1 \wedge y_1 + x_2 \wedge y_2 \),

(ii) \( X = \lambda x_1 \wedge x_2 + y_1 \wedge y_2 \).

PROOF. \( X \in \Lambda^2 V \), \( \ell(X) = 2 \Rightarrow P(X) = 4 \). Extend \( x_1, x_2 \) to a basis \( x_1, x_2, x_3, x_4 \) of \([X]\).

Then \( X = \sum_{1 \leq i < j \leq 4} a_{ij} x_i \wedge x_j, a_{ij} \in F \).
If $a_{34} = 0$, take $y_1 = a_{12}x_2 + a_{13}x_3 + a_{14}x_4$ and $y_2 = a_{23}x_3 + a_{24}x_4$, then $X = x_1 \wedge y_1 + x_2 \wedge y_2$. If $a_{34} \neq 0$, then

$$(-\lambda + a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0 \tag{1})$$

has a solution in $F$.

Set $Y = (-\lambda+a_{12})x_1^x_2 + a_{13}x_3 + a_{14}x_4 + a_{23}x_2^x_3 + a_{24}x_2^x_4 + a_{34}x_3^x_4$.

Then $Y = -\lambda x_1^x_2 + X$. Because of (1), $\ell(Y) = 1$; also $Y \in \Lambda^2[X]$.

Thus $y_1, y_2 \in [X]$ and $y_1 \wedge y_2$. Hence $X = \lambda x_1^x_2 + y_1 \wedge y_2$.

If $X = x_1 \wedge y_1 + x_2 \wedge y_2$ and also $X = \lambda x_1^x_2 + z_1 \wedge z_2$, then $x_1 \wedge X = x_1 \wedge x_2 \wedge y_2$ and also $x_1 \wedge X = x_1 \wedge z_1 \wedge z_2$. Thus $0 \neq x_1 \wedge x_2 \wedge y_2 = x_1 \wedge z_1 \wedge z_2$ and hence $<x_1, x_2, y_2> = <x_1, z_1, z_2>$. Let $z_1 = a_1 x_1 + a_2 x_2 + a_3 y_2$ and $z_2 = b_1 x_1 + b_2 x_2 + b_3 y_2$.

Then $z_1 \wedge z_2 = (a_1 b_2 - a_2 b_1) x_1 \wedge x_2 + (a_1 b_3 - a_3 b_1) x_1 \wedge y_2 + (a_2 b_3 - a_3 b_2) x_2 \wedge y_2$.

Putting this expression for $z_1 \wedge z_2$ in $X = \lambda x_1^x_2 + z_1 \wedge z_2$, we get two different representations of $X$ in the basis of $\Lambda^2[X]$, determined by the basis $x_1, x_2, y_1, y_2$ of $[X]$; thus $X$ has precisely one of the two representations.

**PROPOSITION 10.** If $X, Y \in \Lambda^r Y$ are decomposable, then $X+Y$ is decomposable

iff $\dim[X] \cap [Y] \geq r-1$.

**PROOF.** ($\Rightarrow$) Let $X+Y$ be decomposable, and $X+Y = Z$, $\ell(Z) \leq 1$.

Let $X = x_1 \wedge \ldots \wedge x_r$, $Y = y_1 \wedge \ldots \wedge y_r$, $Z = z_1, \ldots, z_r$. If $[X] = [Z]$, then for any $i$, $1 \leq i \leq r$, $z_i \wedge X = z_i \wedge Z = 0$; but then $z_i \wedge Y = 0$, and thus $z_i \in [Y]$ by Proposition 7, and $[Z] = [Y]$. Hence $[X] = [Y]$, i.e., $\dim[X] \cap [Y] = r$.

If $[X] \neq [Z]$, then for some $i$, $z_i \notin [X]$. But

$z_i \wedge (X+Y) = 0 \Rightarrow z_i \wedge X = -z_i \wedge Y \Rightarrow <z_i, [X]> = <z_i, [Y]>$. Thus $[X], [Y]$ are $r$-dimensional subspaces in an $(r+1) - \dim <z_i, [X]>$. Hence

$\dim[X] \cap [Y] \geq \dim[X] + \dim[Y] - (r+1) = r-1$. ($\Leftarrow$) If $\dim[X] \cap [Y] \geq r-1$.

Let $u_1, \ldots, u_{r-1}$ be l.i. vectors in $[X] \cap [Y]$ and extend these to a basis $x, u_1, \ldots, u_{r-1}$ and a basis $y, u_1, \ldots, u_{r-1}$ of $[X]$ and $[Y]$ respectively. Thus

$X = ax \wedge u_1 \wedge \ldots \wedge u_{r-1}$, $Y = by \wedge u_1 \wedge \ldots \wedge u_{r-1}$ for some $a$ and $b$. 

EQUIVALENCE CLASSES OF THE 3RD GRASSMAN SPACE 303
Hence \(X + Y = (ax + by)^\wedge u_1 \wedge \ldots \wedge u_{r-1}\), i.e., \(X + Y\) is decomposable.

**THEOREM 2.** If \(\dim V = 5\), \(X \in \wedge^3 V\), then \(\ell(X) \leq 2\).

**PROOF.** We shall first prove that \(\ell(X) \leq 3\). Let \(x_1, x_2, x_3, x_4, x_5\) be a basis of \(V\). Then

\[
X = \sum_{1 \leq i < j < k \leq 5} a_{ijk} x_i \wedge x_j \wedge x_k + x_1 \wedge x_2 \wedge (a_{123} x_3 + a_{124} x_4 + a_{125} x_5) + x_1 \wedge x_3 \wedge (a_{134} x_4 + a_{135} x_5) + x_2 \wedge x_3 \wedge (a_{234} x_4 + a_{235} x_5) + (a_{145} x_1 + a_{245} x_2 + a_{345} x_3) x_4 \wedge x_5.
\]

Let \(y_1 = a_{134} x_4 + a_{135} x_5\), \(y_2 = a_{234} x_4 + a_{235} x_5\). If \(y_1, y_2\) are l.d., then \(\ell(X) \leq 3\). So we assume \(y_1, y_2\) are l.i.; then \(\langle y_1, y_2 \rangle = \langle x_4, x_5 \rangle\), and thus \(x_4 \wedge x_5 = \lambda y_1 \wedge y_2, \lambda \in F\). Let \(a_{124} x_4 + a_{125} x_5 = b_1 y_1 + b_2 y_2\). Then

\[
X = x_1 \wedge x_2 \wedge (a_{123} x_3 + b_1 y_1 + b_2 y_2) + x_1 \wedge x_3 \wedge y_1 + x_2 \wedge x_3 \wedge y_2 + \lambda (a_{145} x_1 + a_{245} x_2 + a_{345} x_3) y_1 \wedge y_2
\]

\[
= a_{123} x_1 \wedge x_2 \wedge x_3 + (x_1 + a_{345} \lambda y_2) y_1 \wedge (-b_1 x_2 - x_3 + a_{145} \lambda y_2) + (b_2 x_1 - x_3 - (a_{245} - a_{345} b_1) \lambda y_1) x_2 \wedge y_2.
\]

Hence \(\ell(X) \leq 3\).

Let \(X = X_1 + X_2 + X_3\), where \(X_1, X_2, X_3\) are decomposable, \(X_1 = x_1 \wedge x_2 \wedge x_3\), \(X_2 = y_1 \wedge y_2 \wedge y_3\), \(X_3 = z_1 \wedge z_2 \wedge z_3\). Then \(1 \leq \dim[X_1] \cap [X_2] \leq 3\).

**CASE 1.** \(\dim[X_1] \cap [X_2] = 3\). Then \(X_2 = \lambda X_1\) for some \(\lambda\) and thus \(\ell(X) \leq 2\).

**CASE 2.** \(\dim[X_1] \cap [X_2] = 2\). Let \(u_1, u_2, v, u_1, u_2, w\) be bases of \([X_1]\) and \([X_2]\) respectively. Then \(X_1 = \lambda u_1 \wedge u_2 \wedge v\) and \(X_2 = \lambda u_1 \wedge u_2 \wedge w\). Then \(\ell(X) \leq 2\).

**CASE 3.** \(\dim[X_1] \cap [X_2] = 1\). \(\text{det} u_1, u_2, u_3, u_1, u_4, u_5\) be bases of \([X_1]\) and \([X_2]\) respectively. Then \(X_1 = u_1 \wedge u_2 \wedge u_3, X_2 = u_1 \wedge u_4 \wedge u_5\); we have assumed the co-effs. to be absorbed with the vectors \(u_i\)'s and \(v_i\)'s. Then \(X_1 + X_2 = u_1 \wedge Y\), where \(Y = u_2 \wedge u_3 + u_4 \wedge u_5\). Also \([X_1] + [X_2] = V\).
Since \( \dim \langle u_2, u_3, u_4, u_5 \rangle \cap [X_3] \geq 2 \), we can take \( X_3 = w_1 \wedge w_2 \wedge w_3 \), where

\[ w_1, w_2 \in \langle u_2, u_3, u_4, u_5 \rangle. \]

By Proposition 9, \( v_1, v_2 \) and \( \lambda \ Y = \lambda w_1 \wedge w_2 + v_1 \wedge v_2 \)
or \( Y = w_1 \wedge v_1 + w_2 \wedge v_2 \). If \( Y = \lambda w_1 \wedge w_2 + v_1 \wedge v_2 \), then \( X = u_1 \wedge Y + w_1 \wedge w_2 \wedge w_3 \)
has length \( \leq 2 \). If \( Y = w_1 \wedge v_1 + w_2 \wedge v_2 \), then since \( u_1, w_1, w_2, v_1, v_2 \) is also a basis
of \( V \), let \( w_3 = a_1 u_1 + a_2 w_1 + a_3 w_2 + a_4 v_1 + a_5 v_2 \). Then

\[ X = X_1 + X_2 + X_3 = (u_1 - a_4 w_2) \wedge w_1 \wedge v_1 + u_1 \wedge w_2 \wedge v_2 + (a_5 v_2 + a_1 u_1) \wedge w_1 \wedge w_2 \]
has length \( \leq 2 \), since \( Z = u_1 \wedge w_2 \wedge v_2 + (a_5 v_2 + a_1 u_1) \wedge w_1 \wedge w_2 \)
and

\[ \dim \langle u_1, w_2, v_2, w_1, w_2 \rangle \cap \langle a_5 v_2 + a_1 u_1, w_1, w_2 \rangle \geq 2 \text{ implies } \ell(Z) \leq 1. \]

**REMARK.** There exists \( X \in \wedge^3 V \) with \( \ell(X) = 2 \); for if \( x_1, x_2, x_3, x_4, x_5 \) is
a basis of \( V \) and \( X = x_1 \wedge x_2 \wedge x_3 + x_1 \wedge x_4 \wedge x_5 \), then \( \ell(X) = 2 \), by Proposition 10.

**REMARK.** If \( X \in \wedge^3 V, \dim V = 5, \ell(X) = 2 \), then \( P(X) = 5 \); for let
\( X = X_1 + X_2 \), where \( \ell(X_1) = \ell(X_2) = 1 \). Since \( X \) is not decomposable, then by
Proposition 10, \( \dim[X_1] \cap [X_2] < 2 \) and hence
\[ \dim[X] > \dim[X_1] + \dim[X_2] - \dim[X_1] \cap [X_2] = 4, \text{ i.e., } P(X) = 5. \]

It follows from Proposition 6 that if \( X, Y \in \wedge^3 V \) and \( \ell(X) = \ell(Y) \), then
\( X \sim Y \). Hence all the equivalence classes of \( \wedge^3 V \) are given by

\[ S_0 = \{ X | X \in \wedge^3 V, \ell(X) = 0 \} = \{0\} \]
\[ S_1 = \{ X | X \in \wedge^3 V, \ell(X) = 1 \} \]
\[ S_2 = \{ X | X \in \wedge^3 V, \ell(X) = 2 \}. \]

**ACKNOWLEDGMENT.** The author is grateful to Professor R. Westwick for his
invaluable help in the preparation of this manuscript.

**REFERENCES**

