A NOTE ON Riemann Integrability

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Abstract. In this note we define Riemann integrability for real valued functions defined on a compact metric space accompanied by a finite Borel measure. If the measure of each open ball equals the measure of its corresponding closed ball, then a bounded function is Riemann integrable if and only if its set of points of discontinuity has measure zero.

Let $\mathcal{A}$ denote the algebra of sets generated by the open and closed subintervals of an interval $[a,b]$. A bounded real valued function $f$ defined on $[a,b]$ is Riemann integrable if for each positive $\epsilon$, there exist two functions $\phi$ and $\psi$ that are linear combinations of characteristic functions of sets in $\mathcal{A}$ satisfying $\phi \leq f \leq \psi$ and

$$\int_a^b \psi \ dm - \int_a^b \phi \ dm < \epsilon$$

where $m$ denotes ordinary Lebesgue measure. Riemann integrability may be defined in an analogous way for real valued functions defined on a compact metric space $K$ accompanied by a finite Borel measure. If we make a simple
assumption about the balls of $K$, then the following famous theorem of
Lebesgue extends: a bounded real valued function $f$ defined on $[a,b]$ is
Riemann integrable if and only if the set of points at which $f$ is not
continuous has Lebesgue measure zero.

Suppose that $K$ is a compact metric space and $\mu$ is a finite Borel
measure on $K$. Let $B_r(x) = \{y: d(x,y) < r\}$ and $\overline{B}_r(x) = \{y: d(x,y) \leq r\}$
denote the open and closed balls of radius $r$ about a point $x$ in $K$.
Let $\mathcal{A}$ denote the algebra generated by all such balls. Any element of $\mathcal{A}$
is of the form

$$\bigcup_{1 \leq i \leq m} \bigcap_{1 \leq k \leq n_i} A_{ik}$$

where $A_{ik}$ is a ball or its complement and $\{m,n_1,\ldots,n_m\}$ are positive
integers. A step function is a linear combination of characteristic functions
determined by elements of $\mathcal{A}$. Hence a step function $\phi$ has the form

$$\sum d_i X_{A_i}$$

where each $d_i$ is real and $A_i \in \mathcal{A}$. Since $\mathcal{A}$ is an algebra, the
$\{A_i\}$ can be taken to be pairwise disjoint. It is easy to see that if $\phi$
and $\psi$ are step functions, then so are $\phi + \psi$, $\phi - \psi$, $\inf \{\phi,\psi\}$, and
$\sup \{\phi,\psi\}$.

**DEFINITION.** A bounded real valued function $f$ defined on $K$ is Riemann
integrable if for each positive $\epsilon$ there exist step functions $\phi$ and $\psi$
such that $\phi \leq f \leq \psi$ and $\int \psi \, d\mu - \int \phi \, d\mu < \epsilon$.

Given a bounded real valued function $f$ defined on $K$, the upper
envelope $h$ of $f$ is the function defined by

$$h(x) = \inf_{\delta > 0} \sup_{y \in B_\delta(x)} f(y) \quad x \in K$$
and the lower envelope $g$ of $f$ is defined by

$$g(x) = \sup_{\delta > 0} \inf_{y \in B_\delta(x)} f(y)$$

$x \in K$

It is well known that $h$ is upper semicontinuous, $g$ is lower semicontinuous, $g(x) \leq f(x) \leq h(x)$ for each $x$, and $g(x) = h(x)$ if and only if $f$ is continuous at $x$ (see Royden [1, p.49]).

**THEOREM.** Suppose $u(B_r(x)) = u(\overline{B}_r(x))$ for each $x$ in $K$ and for each positive $r$. A bounded real valued function $f$ defined on $K$ is Riemann integrable if and only if the set of points at which $f$ is discontinuous has $\mu$-measure zero.

**Proof.** Let $h$ be the upper envelope of $f$ and $g$ its lower envelope.

Let $\psi$ be any step function that exceeds $f$. Since each member of $\mathcal{D}$ can be expressed in the form depicted in (1), the condition on the balls of $K$ implies that each member of $\mathcal{D}$ is the union of an open set and a set of $\mu$-measure zero. It follows that $\psi$ can be represented as

$$\sum_{j=1}^{n} a_j x_{A_j}$$

where (i) $A_j$ is an open set for $1 \leq j \leq m$ (ii) $\mu(A_j) = 0$ for $m < j \leq n$ (iii) $\{A_1, A_2, \ldots, A_n\}$ partition $K$.

Let $x \in \bigcup_{j=1}^{m} A_j$. Since $\psi$ is constant near $x$, there exists $\delta > 0$ such that $\psi(x) \geq \sup_{y \in B_\delta(x)} f(y)$ so that $\psi(x) \geq h(x)$. Hence, $\mu(x: \psi(x) < h(x)) = 0$, and we have $\int \psi \, d\mu \geq \int h \, d\mu$. We now construct a decreasing sequence of step functions converging pointwise to $h$ so that
\[ \inf \{ \int \psi \, d\mu : \psi \geq f \text{ and } \psi \text{ is a step function} \} = \int h \, d\mu. \]

Let \( N \) be a fixed positive integer. Let \( \{ B_{r_1}(x_1), \ldots, B_{r_m}(x_m) \} \) be a cover of \( K \) by balls of radius at most \( 1/N \) such that if \( y \in B_{r_i}(x_i), \) then \( h(y) < h(x_i) + 1/N. \) Now let \( \theta_N : K \to \mathbb{R} \) be the step function described by \( \theta_N(x) = \inf \{ h(x_i) + 1/N : x \in B_{r_i}(x_i) \}. \) Define \( \psi_N \) to be \( \theta_N. \) Given any positive integer \( p, \) define \( \theta_{N+p} \) as above, and let \( \psi_{N+p} \) be \( \inf \{ \theta_{N+p}, \psi_{N+p-1} \}. \) Clearly, for each \( p, \psi_{N+p} \) is a step function, and \( \psi_{N+p} \geq \psi_{N+p+1} \geq h. \) To establish the pointwise convergence, suppose to the contrary that for some \( x_0 \) in \( K \) and \( \epsilon > 0 \) we have for each \( p \)

\[ \psi_{N+p}(x_0) > h(x_0) + 2\epsilon. \]

Pick \( n \) so large that \( 1/n < \epsilon. \) There exists a point \( x_n \) such that \( d(x_0, x_n) < 1/n \) and \( \psi_n(x_0) \leq h(x_n) + 1/n.\) Clearly, \( h(x_n) > h(x_0) + \epsilon \) which violates the upper semicontinuity of \( h. \) Hence, \( \{ \psi_n \} \) is the desired sequence.

Using the above technique we can show in the same manner that

\[ \int g \, d\mu = \sup \{ \int \phi \, d\mu : \phi \leq f \text{ and } \phi \text{ is a step function} \}. \]

The proof is now completed by observing the equivalence of the following statements:

(i) \( f \) is Riemann integrable (ii) \( \int g \, d\mu = \int h \, d\mu \) (iii) \( f \) is continuous except at a set of points of \( \mu \)-measure zero.

A simple example shows that the theorem need not hold if our condition on the balls of the metric space is omitted. Let \( K \) be the closed unit disc in the plane with the usual metric. If \( B \) is a Borel subset of \( K, \) define \( \mu(B) \) to be \( \mu_1(B \cap \{(x,y) : x^2 + y^2 = 1\}) + \mu_2(B \cap \{(x,y) : x^2 + y^2 \}

<1) where \( \mu_2 \) is two dimensional Lebesgue measure and \( \mu_1 \) is one dimensional Lebesgue measure, considering the circle as having measure \( 2\pi \). Then the characteristic function of the unit circle is Riemann integrable (being a step function), but its set of discontinuities has measure \( 2\pi \).

REFERENCES


KEY WORDS AND PHRASES. Riemann integrable functions on a compact metric space, Compact metric space with Borel measure.