While trying to solve the problem stated in the introduction of [1] the author became aware of the theorem contained in this paper. Let

\[ a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\
\vdots \\
a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \]  

be a system of linear equations, where each \( a_{ij} \) and \( b_i \) is an integer.

Define a row operation on (1) to be where \( a_{ij}x_j = b_i \) is replaced by

\[ \sum_{j=1}^{n} (a_{ij} + t\alpha_{kj})x_j = b_i + tb_k, \]  

where \( k \neq i \) and \( t \) is an integer.

**THEOREM 1.** Such a system as (1) can be transformed by a finite sequence of row operations into

\[ c_{11}x_1 + \ldots + c_{1n}x_n = d_1 \\
\vdots \\
c_{m1}x_1 + \ldots + c_{mn}x_n = d_m, \]  

(2)

where (1) \( d_2 = \sum_{j=1}^{n} c_{2j} \), (2) \( d_i = \sum_{j=1}^{n} c_{ij} = 0 \) if \( m \geq i > 2 \), and

(3) \( |d_1| < |d_2| \) or \( d_2 = 0 \).

**PROOF.** Given such a system as (1), define \( c_j = \sum_{p=1}^{n} a_{jp} - b_j \) \( (1 \leq j \leq m) \), and define \( E(1) = \sum_{j=1}^{m} |c_j| \). Suppose we consider systems (1), (2), ..., (q) where

\( E(1) > E(2) > \ldots > E(q) \), but no row operation can be performed on (q) to find a
smaller $E$. For convenience suppose the coefficients of $(q)$ are renamed as those in $(1)$.

Suppose there exist two integers $j_1, j_2$ in $[1,m]$ such that $|c_{j_1}| > |c_{j_2}| > 0$. Thus $|c_{j_1}| > |c_{j_2} - c_{j_2}|$ or $|c_{j_1}| > |c_{j_1} + c_{j_2}|$. In the first case replace

$$
\sum_{j=1}^{n} a_{j_1} b_j = b_{j_1} \quad \text{by} \quad \sum_{j=1}^{n} (a_{j_1} b_j - a_{j_2} b_j) x_j = b_{j_1} - b_{j_2},
$$

and in the other case use the sum. In either case the resulting system has a smaller $E$, so in all but one of the equations of $(q)$ the righthand coefficient is the sum of the other coefficients. Using row operations we may suppose row one is the exception.

Still assuming the coefficients of $(q)$ are the $a_{ij}$ and $b_i$ define

$$
D_j = \sum_{p=1}^{n} a_{jp} + b_j \quad \text{and let} \quad N(q) = \sum_{j=1}^{m} |D_j|.
$$

We now suppose $N$ has been reduced as far as possible by row operations which do not involve row one. Since $D_j = 0$ for all but one $j$ in $\{2,\ldots,m\}$, we suppose the exceptional row is two, and that our system is in the form of $(2)$ where conclusions $(1)$ and $(2)$ hold.

To show $(3)$, if $d_2 \neq 0$ and $|d_1| > |d_2|$, we may subtract some integer multiple of row two from row one to produce the desired result.

REFERENCES


KEY WORDS AND PHRASES. Linear equations.

AMS(MOS) SUBJECT CLASSIFICATIONS (1970). 15A06, 10B05.