SOME COUNTEREXAMPLES AND PROPERTIES ON GENERALIZATIONS OF LINDELÖF SPACES

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ABSTRACT. In this paper we give some significative counterexamples to prove that some well known generalizations of Lindelöf property are proper. Also we give some new results, we introduce and study the almost regular-Lindelöf spaces as a generalization of the almost-Lindelöf spaces and as a new and significative application of the quasi-regular open subsets of [1].

KEY WORDS AND PHRASES: Lindelöf space, almost Lindelöf, weakly Lindelöf and nearly Lindelöf, semiregular and almost-regular space, regular cover

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1. INTRODUCTION


In this paper we fix our attention on the main generalizations of Lindelöf spaces, i.e. weakly-Lindelöf, almost-Lindelöf and nearly-Lindelöf spaces. Our purpose is to study the relations between them and some new properties but, mainly, to construct some significative counterexamples to prove that the studied generalizations are proper.

Moreover, the counterexample 3.11, proving that there exist weakly-Lindelöf spaces not almost-Lindelöf, guides us to introduce and study a new generalization of Lindelöf spaces, i.e. the almost regular-Lindelöf spaces. These almost regular-Lindelöf spaces are a new and significative application of quasi-regular open subset introduced by the first author and Lo Faro [1] in 1981.

We conclude the paper proposing the study of two new and natural generalizations of the almost regular-Lindelöf spaces, i.e. the weakly regular-Lindelöf and the nearly regular-Lindelöf spaces.

In particular, this paper is composed of four parts. In §1 we study the nearly-Lindelöf spaces as a generalization of Lindelöf spaces (while Balasubramanian has studied them as a generalization of nearly compact spaces), we give some properties and a counterexample of a nearly-Lindelöf not Lindelöf space. In §2 we study the subspaces and subsets nearly-Lindelöf relative. In §3 we give some properties of
weakly-Lindelof spaces and a counterexample of weakly-Lindelof not nearly-Lindelof space, moreover, we study the almost-Lindelof spaces that are between nearly-Lindelof and weakly-Lindelof spaces, we give the necessary counterexamples and properties. In the last section we introduce the notions of almost regular-Lindelof, weakly regular-Lindelof and nearly regular-Lindelof spaces.

We have that the following implications hold:

$$\text{Nearly-L} \Rightarrow \text{Almost-L} \Rightarrow \text{Weakly-L}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\text{Nearly R-L} \Rightarrow \text{Almost R-L} \Rightarrow \text{Weakly R-L}$$

**PRELIMINARIES**

Throughout the present paper $X$ and $Y$ always denote topological spaces, $x$ an element of $X$ and $\mathcal{U}_x$ the neighborhoods filter of $x$ in $X$. The interior and the closure of any subset $A$ of $X$ will be denoted by $\text{int}(A)$ or $\overline{A}$ and $\text{cl}(A)$ or $\overline{A}$ respectively.

If $A \subseteq S \subseteq X$, then $\text{int}_S(A)$ and $\text{cl}_S(A)$ will be used to denote respectively the interior and closure of $A$ in the subspace $S$. With $\{a_i\}_{i \geq 0}$ and $\{a_i\}_{i \in \mathbb{N}}$ we denote the set of all elements $a_i$ for each $i \geq 0$ and for each $i \in \mathbb{N}$ respectively.

Recall some definitions.

**DEFINITION 1.** A subset $A \subseteq X$ is called **regularly open** (resp. **regularly closed**) if $A = \overline{\text{int}(A)}$ (resp. $A = \overline{A}$).

The topology generated by the regularly open subsets of the space $(X, \tau)$ is denoted by $\tau^*$ and is called **semiregularization** of $X$, if $\tau \equiv \tau^*$ then $X$ is said to be **semiregular** [12].

**DEFINITION 2** [13]. A topological space $X$ is said to be **almost regular** if for each $x \in X$ and each regularly open neighborhood $U_x \in \mathcal{U}_x$, there exists a neighborhood $V_x \in \mathcal{U}_x$ such that $V_x \subseteq \overline{V}_x \subseteq U_x$, or, equivalently, if for any regularly closed set $C$ and any singleton $\{x\}$ disjoint from $C$, there exist disjoint open sets $U$ and $V$ such that $C \subseteq U$ and $x \in V$.

It is true that a space $X$ is regular if and only if it is semiregular and almost regular [13].

**DEFINITION 3** [14]. A topological space $X$ is said to be **nearly compact** if every open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of $X$ admits a finite subfamily such that $X = \bigcup_{\lambda = 1}^{n} \overline{U_\lambda}$.

**DEFINITION 4** [2]. Let $X$ be a topological space. A cover $\mathcal{V} = \{V_j\}_{j \in J}$ of $X$ is a **refinement** of another cover $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ if for each $j \in J$ there exists an $\lambda(j) \in \Lambda$ such that $V_j \subseteq U_{\lambda(j)}$.

**DEFINITION 5** [2]. A family $\{U_\lambda\}_{\lambda \in \Lambda}$ of subsets of a topological space $X$ is **locally finite** if for every point $x \in X$ there exists a neighborhood $U_x \in \mathcal{U}_x$ such that the set $\{\lambda \in \Lambda : U_x \cap U_\lambda \neq \emptyset\}$ is finite.

§1. **NEARLY LINDELÖF-SPACES**

**DEFINITION 1.1** [9]. A topological space $X$ is said to be **nearly-Lindelöf** if for every open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of $X$ there exists a countable subset $\{\lambda_n\}_{n \in \mathbb{N}}$ of $\Lambda$ such that $X = \bigcup_{n \in \mathbb{N}} \overline{U_{\lambda_n}}$ (i.e. if every cover of $X$ by regularly open sets admits a countable subcover).

It is clear that every compact space is nearly-Lindelöf, but the converse is not true (for example the real line $\mathbb{R}$ is nearly-Lindelöf but it is not nearly compact).

Moreover, every Lindelöf space is nearly-Lindelöf but the converse is not true as the following example shows.

**EXAMPLE 1.2.** Let $\Omega$ be the smallest uncountable ordinal number and $A = [0, \Omega)$. The set $A$ has the property that for each $\alpha \in A$ the set $[0, \alpha)$ is countable, while $A$ is not. Let $X = \{a_j, c_i, a\}$ where $i \in A$ and $j \in \mathbb{N}$. We define in $X$ a topology such that the points $\{a_j\}$ are isolated and the fundamental system of neighborhoods of the points $\{c_i\}$ and $\{a\}$ are

$$B_{c_i}^a = \{c_i, a_j\}_{j \geq n} \quad \text{and} \quad B_a^a = \{a, a_j\}_{j \geq 0, j \in \mathbb{N}}$$
respectively \( X \) so topologized is Hausdorff but not Lindelöf, in fact the open cover \( \{ B_1^0 \} \cup \{ B_2^0 \} \) has no countable subcover. On the other hand, \( X \) is nearly-Lindelöf. In fact, let \( \{ U_\omega \} \) be a cover of \( X \) and \( \tilde{X} \) such that \( \alpha \in U_\lambda \). Then \( \left( X \setminus \bigcup \tilde{U}_\lambda \right) \) is a countable set. It follows that \( X \) is nearly-Lindelöf.

**PROPERTY 1.3.** A space \((X, \tau)\) is nearly-Lindelöf if and only if \((X, \tau^*)\) is Lindelöf.

**COROLLARY 1.4.** A nearly-Lindelöf space \((X, \tau)\) is Lindelöf if and only if it is semiregular.

This is an improvement of [Prop 5.5] that holds only for regular spaces.

**PROPOSITION 1.5.** A topological space \( X \) is nearly-Lindelöf if and only if for any family \( \{ C_\lambda \} \) of regularly closed sets of \( X \) with countable intersection property, the intersection \( \bigcap \lambda \in \Lambda C_\lambda \) is non-empty.

**PROPOSITION 1.6.** Let \( X \) be an almost regular and nearly-Lindelöf space. Then for every disjoint regularly closed \( C_1 \) and \( C_2 \) there exist two open sets \( U \) and \( V \) such that \( U \cap V = \emptyset \) and \( C_1 \subseteq U, C_2 \subseteq V \).

**PROOF.** Since \( X \) is almost regular, for each \( x \in C_1 \) there exists an open neighborhood \( U_x \) such that \( U_x \cap C_2 = \emptyset \). We can suppose \( U_x \) to be regularly open. The family \( \{ U_x \} \) is a regularly open cover of \( X \) and, since \( X \) is nearly-Lindelöf, there exists a countable set of points \( x_1, x_2, \ldots, x_n, \ldots \) of \( X \) such that \( X \subseteq \bigcup_{x \in C_1} U_x \). It follows that for each \( n \in \mathbb{N} \), \( C_1 \subseteq \bigcup_{x \in C_1} U_x \).

**DEFINITION 1.7.** A space \( X \) is said to be nearly paracompact if every cover of \( X \) by regularly open sets admits a locally finite refinement.

**PROPOSITION 1.8.** Let \( X \) be an almost regular and nearly-Lindelöf space. Then \( X \) is nearly paracompact.

**PROOF.** Let \( \{ U_\lambda \} \) be a cover of \( X \) by regularly open sets. For each \( x \in X \) and \( \lambda \in \Lambda \) such that \( x \in U_\lambda \), there exists an open neighborhood \( U_x \) of \( x \) such that \( U_x \subseteq U_\lambda \). We can suppose that \( U_x \) is regularly open, so \( \{ U_x \} \) is a regular open cover of \( X \). Since \( X \) is nearly-Lindelöf, there exists a countable set of points \( x_1, x_2, \ldots, x_n, \ldots \) of \( X \) such that \( X \subseteq \bigcup_{x \in C_1} U_x \). For each \( n \in \mathbb{N} \) choose a \( \lambda_n \in \Lambda \) such that \( U_{x_n} \subseteq U_{\lambda_n} \), and put \( V_n = U_{\lambda_n} \setminus \bigcup_{i=1}^{n-1} U_{x_i} \). By construction \( \{ V_n \} \) is a refinement of \( \{ U_\lambda \} \) and it is a locally finite family. In fact, let \( x \in X \). Then there exist \( U_x \) and \( U_{x_n} \) such that \( x \in U_x \subseteq U_{x_n} \). We will prove that \( U_x \) intersects at most finitely many members of the family \( \{ V_n \} \).

**PROPOSITION 1.9.** Let \( X \) be a nearly-Lindelöf space and \( Y \) a nearly compact space. Then \( X \times Y \) is nearly-Lindelöf.

**PROOF.** Let \( \{ U_\lambda \} \) be a cover of \( X \times Y \) by regularly open sets. Without loss of generality, we can suppose \( U_\lambda = V_\lambda \times W_\lambda \) where \( V_\lambda \) and \( W_\lambda \) are regularly open sets in \( X \) and \( Y \) respectively. Fix \( x \in X \), for each \( y \in Y \) there exists \( \lambda_y \in \Lambda \) such that \((x, y) \in V_{\lambda_y} \times W_{\lambda_y} \). The family \( \{ W_{\lambda_y} \}_{y \in Y} \) is a cover of \( Y \) by regularly open sets and, since \( Y \) is nearly compact, it admits a finite subcover. Let \( Y = W_{\lambda_1} \cup \ldots \cup W_{\lambda_y} \). Put \( H_x = V_{\lambda_1} \cap \ldots \cap V_{\lambda_y} \) and \( \tau \) of \( \{ \lambda_1, \ldots, \lambda_y \} \). \( H_x \) is a regularly open set of \( X \) and hence the family \( \{ H_x \}_{x \in X} \) is a regularly open cover of \( X \). Since \( X \) is nearly-Lindelöf, there exists a countable set of points \( x_1, x_2, \ldots, x_n, \ldots \) of \( X \) such that \( X = \bigcup_{n \in \mathbb{N}} H_{x_n} \), hence...
$X \times Y = \left( \bigcup_{n \in \mathbb{N}} H_n \right) \times Y = \bigcup_{n \in \mathbb{N}, \tau_{(x_n)}} (H_n \times W_n) = \bigcup_{n \in \mathbb{N}, \tau_{(x_n)}} (V_n \times W_n)$.

Since the last member is a countable family, we have that $X \times Y$ is nearly-Lindelöf.

**Remark 1.10** In general the product of two nearly-Lindelöf spaces is not nearly-Lindelöf. In fact, let $K$ be the Sorgenfrey line $K$ is normal, and hence regular, and Lindelöf and therefore it is nearly-Lindelöf. The product $K \times K$ is regular, but it is not Lindelöf [2, 38 15] and therefore it cannot be nearly-Lindelöf (see Corollary 1.4).

**§ 2. NEARLY-LINDELÖF SUBSPACES AND SUBSETS**

A subset $S$ of a space $X$ is said to be nearly-Lindelöf if $S$ is nearly-Lindelöf as subspace of $X$ (i.e. $S$ is nearly-Lindelöf with respect to the induced topology from the topology of $X$).

**Definition 2.1** A subset $S$ of a space $X$ is said to be **nearly-Lindelöf relative to $X$** if for every cover $\{U_\lambda\}_{\lambda \in \Lambda}$ by open sets of $X$ such that $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$, there exists a countable subset $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \Lambda$ such that $S \subseteq \bigcup_{n \in \mathbb{N}} U_{\lambda_n}$.

**Proposition 2.2** [9] Let $X$ be a space and $A$ an open subset of $X$. Then $A$ is nearly-Lindelöf if and only if it is nearly-Lindelöf relative to $X$.

**Lemma 2.3** [9] Let $B$ be a regularly closed subset of a nearly-Lindelöf space $X$. Then $B$ is nearly-Lindelöf relative to $X$.

**Corollary 2.4** [9] A clopen of a nearly-Lindelöf space $X$ is nearly-Lindelöf.

**Property 2.5** Let $X$ be an extremally disconnected space (i.e., the closure of an open set is open [2]) and $S \subseteq X$. If $S$ is nearly-Lindelöf then $S$ is nearly-Lindelöf relative to $X$.

**Proof.** Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open family of $X$ such that $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$. Consider $V_\lambda = S \cup U_\lambda$ for each $\lambda \in \Lambda$, then $\{V_\lambda\}_{\lambda \in \Lambda}$ is an open cover of $S$. By hypothesis there exists a countable subfamily $\{V_{\lambda_n}\}_{n \in \mathbb{N}}$ such that $S \subseteq \bigcup_{n \in \mathbb{N}} V_{\lambda_n}$. Since for each $n \in \mathbb{N}$, $V_{\lambda_n} \subseteq U_{\lambda_n}$, then $V_{\lambda_n} \subseteq U_{\lambda_n}$. Since $X$ is extremally disconnected then $\text{int}_c \text{cl}_c (V_{\lambda_n}) \subseteq \text{int}_c \text{cl}_c (U_{\lambda_n}) = \text{cl}_c (U_{\lambda_n})$. This proves that $S \subseteq \bigcup_{n \in \mathbb{N}} U_{\lambda_n}$, i.e. $S$ is nearly-Lindelöf relative to $X$.

**Remark 2.6.** In general a closed subset of a nearly-Lindelöf space is neither nearly-Lindelöf nor nearly-Lindelöf relative to the space as the subset $\{c, a\}$ in Example 1.2 shows.

**Proposition 2.7.** Let $X$ be a space and $c \subseteq X$. The following are equivalent:

(i) $S$ is nearly-Lindelöf relative to $X$;

(ii) for every family by regularly open sets of $X$ that cover $S$, there exists a countable subfamily covering $S$;

(iii) for every family $\{C_\lambda\}_{\lambda \in \Lambda}$ by regularly closed sets of $X$ such that $\left( \bigcap_{\lambda \in \Lambda} C_\lambda \right) \cap S = \emptyset$, there exists a countable subset of indices $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \Lambda$ such that $\left( \bigcap_{n \in \mathbb{N}} C_{\lambda_n} \right) \cap S = \emptyset$.

**Proof.** (i) $\Rightarrow$ (ii) It is obvious by the definition.

(ii) $\Rightarrow$ (iii). Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a regularly closed family in $X$ such that $\left( \bigcap_{\lambda \in \Lambda} C_\lambda \right) \cap S = \emptyset$. Then $S \subseteq X \setminus \left( \bigcap_{\lambda \in \Lambda} C_\lambda \right) = \bigcup_{\lambda \in \Lambda} (X \setminus C_\lambda)$; hence $\{X \setminus C_\lambda\}_{\lambda \in \Lambda}$ is a regularly open family covering $S$, then there exists a countable subfamily $\{X \setminus C_{\lambda_n}\}_{n \in \mathbb{N}}$ such that $S \subseteq \bigcup_{n \in \mathbb{N}} (X \setminus C_{\lambda_n})$, i.e $\left( \bigcap_{n \in \mathbb{N}} C_{\lambda_n} \right) \cap S = \emptyset$.

(iii) $\Rightarrow$ (i) Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a family by open subsets of $X$ such that $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$. Then $S \subseteq B \cup_{\lambda \in \Lambda} U_\lambda \subseteq \bigcup_{\lambda \in \Lambda} U_{\lambda_n}$, therefore $\left( X \setminus \left( \bigcup_{\lambda \in \Lambda} U_\lambda \right) \right) \cap S = \emptyset$, i.e $\left( \bigcup_{\lambda \in \Lambda} U_{\lambda_n} \right) \cap S = \emptyset$. By hypothesis there exists a countable subfamily $\{X \setminus U_{\lambda_n}\}_{n \in \mathbb{N}}$ such that $\left( \bigcap_{n \in \mathbb{N}} (X \setminus U_{\lambda_n}) \right) \cap S = \emptyset$ and therefore $\left( X \setminus \left( \bigcup_{n \in \mathbb{N}} U_{\lambda_n} \right) \right) \cap S = \emptyset$, i.e $S \subseteq \bigcup_{n \in \mathbb{N}} U_{\lambda_n}$. This completes the proof.
PROPOSITION 2.8. A space \((X, \tau)\) is open hereditarily nearly-Lindelof if and only if any \(A \in \tau^*\) is nearly-Lindelof

**Proof.** Let \(B \subset X\) be an open subset of \(X\). By Proposition 2.2 it is sufficient to prove that \(B\) is nearly-Lindelof relative to \(X\). Let \(\{U_\lambda\}_{\lambda \in \Lambda}\) be a family by regularly open sets of \(X\) such that \(B \subset \bigcup_{\lambda \in \Lambda} U_\lambda\). The set \(A = \bigcup_{\lambda \in \Lambda} U_\lambda\) belongs to \(\tau^*\), so by hypothesis \(A\) is nearly-Lindelof. Hence there exists a countable subfamily \(\{U_{\lambda_n}\}_{n \in \mathbb{N}}\) of \(\{U_\lambda\}_{\lambda \in \Lambda}\) such that \(A = \bigcup_{n \in \mathbb{N}} U_{\lambda_n}\) and therefore \(B \subset \bigcup_{n \in \mathbb{N}} U_{\lambda_n}\). Conversely, let \(X\) be open hereditarily nearly-Lindelof. Since \(\tau^* \subset \tau\), it is obvious that any \(A \in \tau^*\) is nearly-Lindelof. □

**Theorem 2.9.** Let \(f: X \to Y\) be a closed continuous function and, for each \(V \subset Y\), let \(f^{-1}(V)\) be nearly-Lindelof relative to \(X\). If \(Y\) is nearly-Lindelof then \(X\) is nearly-Lindelof.

**Proof.** Let \(\{C_\mu\}_{\mu \in M}\) be a family of regularly closed subsets of \(X\) with countable intersection property. Let \(M = \Lambda^N\), i.e., each \(\mu \in M\) is of the form \(\mu = (\lambda_1, \lambda_2, ..., \lambda_n, ...)\). Put \(C_\mu = \bigcap_{n \in \mathbb{N}} C_{\lambda_n} \neq \emptyset\). The family \(\{C_\mu\}_{\mu \in M}\) is a family by closed subsets of \(X\) with countable intersection property and also the family \(\{f(C_\mu)\}_{\mu \in M}\) in \(Y\) is so. Since \(Y\) is nearly-Lindelof, by Proposition 1.5 there exists \(\bar{y} \in Y\) such that \(\bar{y} \in f(C_\mu)\) for each \(\mu \in M\). It follows that \(f^{-1}(\bar{y}) \cap C_\mu \neq \emptyset\) for each \(\mu \in M\), hence \(f^{-1}(\bar{y})\) intersects all countable intersections of \(C_\lambda\) with \(\lambda \in \Lambda\). Since \(f^{-1}(\bar{y})\) is nearly-Lindelof relative to \(X\), by Proposition 2.7 (iii), we have \(\bigcap_{\lambda \in \Lambda} C_\lambda \cap f^{-1}(\bar{y}) \neq \emptyset\) and thus \(\bigcap_{\lambda \in \Lambda} C_\lambda \neq \emptyset\). This, by Proposition 1.5, implies that \(X\) is nearly-Lindelof. □

**Remark 2.10.** Recall that, for a topological space \(X\), the Lindelof number \(l(X)\) is defined as the smallest cardinal number \(\lambda\) such that every open cover of \(X\) admits a subcover of cardinality \(\lambda\). It is natural to generalize this notion to nearly-Lindelof space defining the nearly-Lindelof number of \(X\) \(nl(X)\) to be the smallest cardinal number \(\mu\) such that every regularly open cover of \(X\) admits a subcover of cardinality \(\mu\).

Obviously \(nl(X) \leq l(X)\) and this inequality can be proper. For this purpose we can see Example 1.2.

§3. ALMOST-LINDELOF AND WEAKLY-LINDELOF SPACES

**Definition 3.1.** [10] A topological space \(X\) is called almost-Lindelof if every open cover \(\{U_\lambda\}_{\lambda \in \Lambda}\) of \(X\) admits a countable subfamily such that \(X \setminus \bigcup_{\lambda \in \Lambda} U_\lambda = \emptyset\).

**Definition 3.2.** [3] A topological space \(X\) is said to be weakly-Lindelof if for every open cover \(\{U_\lambda\}_{\lambda \in \Lambda}\) of \(X\) there exists a countable subfamily such that \(X = \bigcup_{\lambda \in \Lambda} U_\lambda\).

**Proposition 3.3.** A topological space \(X\) is weakly-Lindelof if and only if for any family of closed subsets of \(X\) \(\{C_\lambda\}_{\lambda \in \Lambda}\) such that \(\bigcap_{\lambda \in \Lambda} C_\lambda = \emptyset\) there exists a countable subfamily \(\{C_{\lambda_n}\}_{n \in \mathbb{N}}\) such that \(\operatorname{int} \left(\bigcap_{n \in \mathbb{N}} C_{\lambda_n}\right) = \emptyset\).

**Proof.** Let \(\{C_\lambda\}_{\lambda \in \Lambda}\) be a family of closed subsets of \(X\) such that \(\bigcap_{\lambda \in \Lambda} C_\lambda = \emptyset\). Then \(X = \bigcup_{\lambda \in \Lambda} (X \setminus C_\lambda)\), so by hypothesis there exists a countable subfamily such that \(X = \bigcup_{n \in \mathbb{N}} (X \setminus C_{\lambda_n})\). Hence \(X \setminus \bigcup_{n \in \mathbb{N}} (X \setminus C_{\lambda_n}) = \emptyset\) i.e., \(\operatorname{int} \left(\bigcap_{n \in \mathbb{N}} (X \setminus C_{\lambda_n})\right) = \operatorname{int} \left(\bigcap_{n \in \mathbb{N}} C_{\lambda_n}\right) = \emptyset\). Conversely, let \(\{U_\lambda\}_{\lambda \in \Lambda}\) be an open cover of \(X\). Then \(\bigcap_{\lambda \in \Lambda} (X \setminus U_\lambda) = \emptyset\) and therefore there exists a countable subfamily such that \(\operatorname{int} \left(\bigcap_{n \in \mathbb{N}} (X \setminus U_{\lambda_n})\right) = \emptyset\). So \(X = X \setminus \operatorname{int} \left(\bigcap_{n \in \mathbb{N}} (X \setminus U_{\lambda_n})\right) = X \setminus \left(\bigcap_{n \in \mathbb{N}} (X \setminus U_{\lambda_n})\right) = \bigcup_{n \in \mathbb{N}} U_{\lambda_n}\). □

**Proposition 3.4.** Let \(X\) be a topological space. For the following conditions

(i) \(X\) is weakly-Lindelof,
(ii) any cover \( \{U_\lambda\}_{\lambda \in \Lambda} \) of \( X \) by regularly open sets of \( X \) admits a countable subfamily with dense union in \( X \),

(iii) if \( \{C_\alpha\}_{\alpha \in \Lambda} \) is a family of regularly closed subsets of \( X \) such that \( \bigcap_{\alpha \in \Lambda} C_\alpha = \emptyset \), then there exists a countable subfamily such that \( \operatorname{int} \left( \bigcap_{n \in N} C_{\lambda_n} \right) = \emptyset \),

we have that (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (ii) and if \( X \) is semiregular then (ii) \( \Rightarrow \) (i).

**Proof.** (i) \( \Rightarrow \) (ii) is obvious from the definition. The proof of (ii) \( \Rightarrow \) (iii) is quite similar to the proof of Proposition 3.3 replacing open cover with a regularly open cover of \( X \). We will prove the implication (ii) \( \Rightarrow \) (i) when \( X \) is semiregular. Let \( \{U_\lambda\}_{\lambda \in A} \) be an open cover of \( X \). By hypothesis we can suppose any \( U_\lambda \) to be regularly open. Then there exists a countable subfamily \( \{U_{\lambda_n}\}_{n \in N} \) such that \( \bigcup_{n \in N} U_{\lambda_n} = X \). This completes the proof.

Obviously, if a space is nearly-Lindelöf then it is almost-Lindelöf and if a space is almost-Lindelöf then it is weakly-Lindelöf. But the following example shows that weakly-Lindelöf property or almost-Lindelöf property does not imply the nearly-Lindelöf property.

**Example 3.5.** Let \( \Omega \) be the smallest uncountable ordinal number and \( A = [0, \Omega) \) as in Example 1.2. Let \( X = \{a_0, b, c, d, \} \) where \( i \in A \) and \( j \in \mathbb{N} \). Consider in \( X \) the topology such that the points \( \{a_i\} \) and \( \{b_j\} \) are isolated and the fundamental system of neighborhoods of the points \( \{c_i\}, \{a\} \) and \( \{b\} \) are

\[
B_n = \{c_i, a_j, b_j\}_{j \geq n}, \quad B_a = \{a_i, a_j\}_{i \geq 0, j \in \mathbb{N}} \quad \text{and} \quad B_b = \{b_i, b_j\}_{i \geq 0, j \in \mathbb{N}}
\]

respectively. \( X \) so topologized is Hausdorff and semiregular but it is not nearly-Lindelöf as we can see considering the regularly open cover \( \left( \bigcup_{n \in N} B_n \bigcup B_a \bigcup B_b \right) \). But \( X \) is weakly-Lindelöf. Indeed, let \( \{U_\lambda\}_{\lambda \in A} \) be an open cover of \( X \). Then there exist \( \lambda_1, \lambda_2 \in \Lambda \) such that \( a \in U_{\lambda_1} \) and \( b \in U_{\lambda_2} \). The set \( X \setminus (U_{\lambda_1} \cup U_{\lambda_2}) \) is countable, so it follows easily that \( X \) is weakly-Lindelöf. Note that this space \( X \) is also almost-Lindelöf.

Below we will give the construction of an example of a weakly-Lindelöf space that it is not almost-Lindelöf.

**Proposition 3.6.** A topological space \( X \) is almost-Lindelöf if and only if every family \( \{C_\alpha\}_{\alpha \in \Lambda} \) of closed subsets of \( X \) such that \( \bigcap_{\alpha \in \Lambda} C_\alpha = \emptyset \) admits a countable subfamily such that \( \bigcap_{n \in N} C_{\lambda_n} = \emptyset \).

**Proof.** If \( \{C_\alpha\}_{\alpha \in \Lambda} \) is a family by closed subsets of \( X \) such that \( \bigcap_{\alpha \in \Lambda} C_\alpha = \emptyset \), then the family \( \{X \setminus C_\alpha\}_{\alpha \in \Lambda} \) is an open cover of \( X \). By hypothesis there exists a countable subfamily such that \( \bigcup_{n \in N} X \setminus C_{\lambda_n} = X \). Conversely, let \( \{U_\lambda\}_{\lambda \in \Lambda} \) be an open cover of \( X \). Then \( X \setminus \{X \setminus U_\lambda\}_{\lambda \in \Lambda} \) is a family by closed sets such that \( \bigcap_{n \in N} (X \setminus U_{\lambda_n}) = \emptyset \). By hypothesis there exists a countable subfamily such that \( \bigcap_{n \in N} (X \setminus U_{\lambda_n}) = \emptyset \). This completes the proof.

**Proposition 3.7.** Let \( X \) be a topological space. For the following conditions

(i) \( X \) is almost-Lindelöf,

(ii) every regularly open cover \( \{U_\lambda\}_{\lambda \in \Lambda} \) admits a countable subfamily such that \( X = \bigcup_{n \in N} U_{\lambda_n} \),

(iii) every family \( \{C_\alpha\}_{\alpha \in \Lambda} \) of regularly closed subsets of \( X \) such that \( \bigcap_{\alpha \in \Lambda} C_\alpha = \emptyset \) admits a countable subfamily such that \( \bigcap_{n \in N} C_{\lambda_n} = \emptyset \),

we have that (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (ii) and if \( X \) is semiregular then (ii) \( \Rightarrow \) (i).

**Proof.** (i) \( \Rightarrow \) (ii) is obvious by the definition. The proof of (ii) \( \Leftrightarrow \) (iii) is quite similar to the proof of Proposition 3.6 replacing open cover with a regularly open cover of \( X \). We will prove the implication (ii) \( \Rightarrow \) (i) when \( X \) is semiregular. Let \( \{U_\lambda\}_{\lambda \in \Lambda} \) be an open cover of \( X \). By hypothesis we can suppose that any \( U_\lambda \) is regularly open, then there exists a countable subfamily \( \{U_{\lambda_n}\}_{n \in N} \) such that \( \bigcup_{n \in N} U_{\lambda_n} = X \). This completes the proof.
THEOREM 3.8. A weakly-Lindelof, semiregular and nearly paracompact space $X$ is almost-Lindelof

PROOF. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a cover of $X$ by regularly open sets. Since $X$ is nearly paracompact, this cover admits a locally finite refinement $\{V_\alpha\}_{\alpha \in \Gamma}$. Since $X$ is weakly-Lindelof then there exists a countable subfamily such that $X = \bigcup_{\alpha \in \Gamma} V_\alpha$. Since the family $\{V_\alpha\}_{\alpha \in \Gamma}$ is locally finite, then $\bigcup_{\alpha \in \Gamma} V_\alpha = X = \bigcup_{\alpha \in \Gamma} V_\alpha [2, 11 11]$. Choosing, for each $n \in \mathbb{N}$, $\lambda_n \in \Lambda$ such that $V_\alpha \subseteq U_{\lambda_n}$, we obtain $X = \bigcup_{n \in \mathbb{N}} V_{\lambda_n} = \bigcup_{n \in \mathbb{N}} U_{\lambda_n}$. By Proposition 3.7 $X$ is almost-Lindelof. □

PROPOSITION 3.9 [9] An almost regular space is an almost-Lindelof space if and only if it is nearly-Lindelof. □

CONSTRUCTION OF A WEAKLY-LINDELOF SPACE

LEMMA 3.10. The real line $\mathbb{R}$ can be partitioned in the union of a family, of cardinality $2^{\aleph_0}$, by countable dense and pairwise disjoint subsets of $\mathbb{R}$

PROOF. Let $Q$ be the set of the rational numbers. Consider the following equivalence relation on $\mathbb{R}$

$x \sim y$ if and only if $x - y \in Q$.

The partition of $\mathbb{R}$ so obtained is the one that we want, in fact every equivalence class is of the form $x + Q$, where $x \in \mathbb{R}$, and it is a countable dense subset of $\mathbb{R}$. □

EXAMPLE 3.11. Let $\mathbb{R}$ be the real line and $\tau$ the usual topology on it. By the previous lemma we can represent $\mathbb{R}$ as a union of a family, of cardinality $2^{\aleph_0}$, by countable dense and pairwise disjoint subsets of $\mathbb{R}$.

Consider the open cover of $X$

$X = \bigcup_{i \in I} S_i$.

Suppose that $X$ is almost-Lindelof, then there exists a countable set $\{i_1, i_2, \ldots, i_n, \ldots\} \subseteq I$ such that

$X = \bigcup_{n \in \mathbb{N}} S_{i_n} \cup \bigcup_{n \in \mathbb{N}} \left[ x_n - \frac{1}{n^2}, x_n + \frac{1}{n^2} \right]$.

Since the Lebesgue measure of the set $\bigcup_{n \in \mathbb{N}} \left[ x_n - \frac{1}{n^2}, x_n + \frac{1}{n^2} \right]$ is finite, then $X \subseteq \bigcup_{n \in \mathbb{N}} \left( S_{i_n} \cup S_0 \right)$ and, since the second member is countable, we obtain a contradiction. We will show now that $X$ is weakly-Lindelof. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $X$ and $S_0 = \{x_0, x_1, \ldots, x_n, \ldots\}$ as above. Since in the topology $\sigma$ every point of $S_0$ has the same fundamental system of neighborhoods as in the topology $\tau$, then for each $n \in \mathbb{N}$ there exist an open set $V_n$ in $\tau$ and an index $\lambda_n \in \Lambda$ such that $x_n \in V_n \subseteq U_{\lambda_n}$. The set $V = \bigcup_{n \in \mathbb{N}} V_n$ is open in $\tau$ and $S_0 \subseteq V \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$. For any $\tau$-open neighborhood $V_1$ of $x_1$, it is $V_1 \cap S_0 = \emptyset$ (because $S_0$ is dense in $(\mathbb{R}, \tau)$). So $V_1 \cap V = \emptyset$, hence $S_1 \cap V_1 \cap V = \emptyset$ and this shows that $x_1 \in \text{cl}_\sigma(V)$.

We obtain that $X = \text{cl}_\sigma(V) = \text{cl}_\sigma \left( \bigcup_{\lambda \in \Lambda} U_\lambda \right)$ and therefore $X$ is weakly-Lindelof. □

§4. ALMOST REGULAR-LINDELOF SPACES

The previous example suggests some interesting remarks. But before it is useful to recall the following definitions

DEFINITION 4.1 [1] An open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of a topological space $X$ is said to be regular if for every $\lambda \in \Lambda$ there exists a non-empty regularly closed subset $C_\lambda$ of $X$ such that $C_\lambda \subseteq U_\lambda$ (i.e. $U_\lambda$ is quasi regular open) and $\bigcup_{\lambda \in \Lambda} C_\lambda = X$. 

GENERALIZATIONS OF LINDELOF SPACES 743
**Definition 4.2** [1] A topological space $X$ is said to be weakly compact if every regular cover admits a finite subfamily such that the union is dense in $X$.

**Lemma 4.3.** Let $X$ be the space in Example 3.11. Let $C$ be a regularly closed and $A$ an open set such that $C \subset A$. Then $\text{int}_\sigma(C) \subset \text{int}_\tau(C)$.

**Proof.** We denote with $x_0$ and $x_1$ the elements of $S_0$ and $S$, respectively. We show before that if $x_0 \in \text{int}_\sigma(C)$ then $x_0 \in \text{int}_\tau(C) \subset \text{int}_\tau(A)$ Since the fundamental system of neighborhoods of $x_0$ is the same in the topology $\sigma$ or in $\tau$, then the lemma is true. Now let $x_i \in \text{int}_\sigma(C)$ There exists a $\tau$-open neighborhood $V_i$ of $x_i$ such that $V_i \cap S \subset C$. We will show that $V_i \subset \text{cl}_\tau(A)$ Let $x_i \in V_i$, and let $V_0$ be an arbitrary $\sigma$-open, and therefore $\tau$-open, neighborhood of $x_0$. Since $x_0 \in V_i \cap V_0$, we have $V_i \cap V_0 \neq \emptyset$ and thus $S_i \cap V_i \cap V_0 \neq \emptyset$. This shows that $x_0 \in \text{cl}_\tau(V_i \cap S_i \subset C \subset \text{cl}_\tau(A)$ Let $x_j \in V_j$. Suppose that $x_j \not\in \text{cl}_\tau(A)$, i.e. there exists a $\tau$-open neighborhood $V_j'$ of $x_j$ such that $V_j \cap S_j \cap A = \emptyset$. Since $x_j \in V_j \cap V_0$, then $V_j \cap V_0 \neq \emptyset$ and therefore $V_j \cap V_0 \cap V_0 \neq \emptyset$. Let $x_0 \in V_j \cap V_0$. We have seen above that $x_0 \in C \subset A \subset \text{cl}_\tau(A)$, since $A$ is $\sigma$-open hence there exists a $\sigma$-open, and therefore $\tau$-open neighborhood $V_0$ of $x_0$ such that $V_0 \subset A$. Since $V_0 \cap V_j \cap V_0 \neq \emptyset$, then, by density of $S_j$, $V_0 \cap V_j \cap V_j \neq \emptyset$ and therefore $A \cap V_j \cap V_j \neq \emptyset$ that is a contradiction. So it is shown that $x_j \in \text{cl}_\tau(A)$ and therefore $V_j \subset \text{cl}_\tau(A)$. The proof is complete.

**Property 4.5.** The space $X$ in Example 3.11 satisfies the following property. Every regular cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of $X$ admits a countable subfamily $\{U_{\lambda_\alpha}\}_{\alpha \in \Lambda}$ such that $X = \bigcup_{\alpha \in \Lambda} U_{\lambda_\alpha}$

**Proof.** Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a regular cover of $X$. For any $\lambda \in \Lambda$ there exists a regularly closed $C_\lambda \subset U_\lambda$ such that $X = \bigcup_{\lambda \in \Lambda} \text{int}_\sigma(C_\lambda)$. By the previous lemma we have $X = \bigcup_{\lambda \in \Lambda} \text{int}_\tau(C_\lambda)$ and, since $X$ is Lindelof with respect to the topology $\tau$, there exists a countable subcover such that $X = \bigcup_{\alpha \in \Lambda} \text{int}_\tau(C_\lambda)$. The property is complete.

**Definition 4.6.** A topological space $X$ is called almost regular-Lindelof if every regular cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of $X$ admits a countable subfamily such that $X = \bigcup_{\lambda \in \Lambda} U_{\lambda}$.

**Remark 4.7.** Obviously almost-Lindelof implies almost regular-Lindelof, but the converse in general is not true, in fact the space $X$ in Example 3.11 is almost regular-Lindelof but not almost-Lindelof.

**Theorem 4.8.** An almost regular-Lindelof and almost regular space $X$ is nearly-Lindelof.

**Proof.** Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a cover by regularly open sets of $X$. For each $x \in X$ there exists $\lambda_x \in \Lambda$ such that $x \in U_{\lambda_x}$. Since $X$ is Lindelof with respect to the topology $\tau$, there exist two regularly open subsets $V_\lambda$ and $W_\lambda$ such that $x \in V_\lambda \subset W_\lambda \subset C_{\lambda_x} \subset U_{\lambda_x}$ [9, Th 2.2]. The family $\{W_\lambda\}_{x \in X}$ is a regular cover of $X$ and, since $X$ is almost regular-Lindelof, there exists a countable set of points $x_1, x_2, \ldots, x_n, \ldots$ of $X$ such that $X = \bigcup_{x \in X} W_\lambda$. So $X = \bigcup_{\lambda \in \Lambda} U_{\lambda}$, and therefore $X$ is nearly-Lindelof.

The previous theorem implies the following:

**Corollary 4.9.** Let $X$ be an almost regular space. Then $X$ is almost regular-Lindelof if and only if it is nearly-Lindelof.

We give now a characterization of almost regular-Lindelof spaces:

**Theorem 4.10.** A topological space $X$ is almost regular-Lindelof if and only if for every family $\{C_\lambda\}_{\lambda \in \Lambda}$ of closed subsets of $X$, such that for each $\lambda \in \Lambda$ there exists an open set $A_\lambda \supset C_\lambda$ with $\bigcap_{\lambda \in \Lambda} A_\lambda = \emptyset$, there exists a countable subfamily such that $\bigcap_{\lambda \in \Lambda} C_\lambda = \emptyset$.

**Proof.** Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a family of closed sets of $X$ such that for each $\lambda \in \Lambda$ there exists an open set $A_\lambda \supset C_\lambda$ with $\bigcap_{\lambda \in \Lambda} A_\lambda = \emptyset$. It follows that $X = \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda \in \Lambda} (X \setminus A_\lambda)$. But, since $C_\lambda \subset A_\lambda \subset A_\lambda \subset \bigcup_{\lambda \in \Lambda} A_\lambda$, then $X \setminus A_\lambda \subset X \setminus A_\lambda \subset X \setminus C_\lambda$, and therefore $x = \bigcup_{\lambda \in \Lambda} (X \setminus C_\lambda)$.

The family...
\{X \setminus C_{\lambda}\}_{\lambda \in \Lambda} is a regular cover of \(X\), since \(X\) is almost regular-Lindelöf, then there exists a countable subfamily such that
\[ X = \bigcup_{n \in \mathbb{N}} \left( X \setminus \left( \bigcap_{\lambda \in \Lambda} \hat{C}_{\lambda} \right) \right) = \bigcup_{n \in \mathbb{N}} \left( X \setminus \left( \bigcap_{\lambda \in \Lambda} \hat{C}_{\lambda} \right) \right) \]
and therefore \( \bigcap_{\lambda \in \Lambda} \hat{C}_{\lambda} = \emptyset \). Conversely, let \( \{U_\lambda\}_{\lambda \in \Lambda} \) be a regular cover of \(X\). For each \(\lambda \in \Lambda\) there exists a regularly closed \(C_\lambda\) of \(X\) such that \(\hat{C}_\lambda \subseteq C_\lambda \subseteq U_\lambda\) and \(\bigcup_{\lambda \in \Lambda} \hat{C}_\lambda = X\). The family \(\{X \setminus U_\lambda\}_{\lambda \in \Lambda}\) of closed sets is such that, for each \(\lambda \in \Lambda\), there exists the open set \(X \setminus C_\lambda \supset X \setminus U_\lambda\) and such that
\[ \bigcap_{\lambda \in \Lambda} \left( X \setminus C_{\lambda} \right) = \bigcap_{\lambda \in \Lambda} \left( X \setminus C_{\lambda} \right) = X \setminus X = \emptyset. \]
By hypothesis, there exists a countable set of indices \(\{\lambda_n\}_{n \in \mathbb{N}}\) such that \(\bigcap_{n \in \mathbb{N}} \operatorname{int}(X \setminus U_{\lambda_n}) = \emptyset\), i.e.
\[ \bigcap_{n \in \mathbb{N}} \left( X \setminus \bigcup_{\lambda \in \Lambda} U_{\lambda} \right) = \emptyset \]So \(X \setminus \left( \bigcup_{n \in \mathbb{N}} U_{\lambda_n} \right) = \emptyset\) and therefore \(X = \bigcup_{n \in \mathbb{N}} U_{\lambda_n}\). This completes the proof.

**ALMOST REGULAR-LINDELÖF SUBSPACES AND SUBSETS**

A subset \(S\) of a space \(X\) is said to be almost regular-Lindelöf if \(S\) is almost regular-Lindelöf as a subspace of \(X\).

**DEFINITION 4.11.** A subset \(S\) of a space \(X\) is said to be almost regular-Lindelöf relative to \(X\) if for each family \(\{U_\lambda\}_{\lambda \in \Lambda}\) of open sets of \(X\) satisfying the condition
\[ S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda, \]
and
\[ \left( \ast \right) \text{ for each } \lambda \in \Lambda, \text{ there exists a nonempty regularly closed set } C_\lambda \text{ of } X \text{ such that } C_\lambda \subseteq U_\lambda \text{ and } S \subseteq \bigcup_{\lambda \in \Lambda} \hat{C}_\lambda, \]
there exists a countable subset \(\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \Lambda\) such that \(S \subseteq \bigcup_{n \in \mathbb{N}} U_{\lambda_n}\).

**THEOREM 4.12.** If \(S\) is an almost regular-Lindelöf subspace of a space \(X\), then \(S\) is almost regular-Lindelöf relative to \(X\).

**PROOF.** Let \(\{U_\lambda\}_{\lambda \in \Lambda}\) be a cover of \(S\) satisfying the condition \(\left( \ast \right)\). For each \(\lambda \in \Lambda\), we have that \(\hat{C}_\lambda \cap S \subseteq U_\lambda \cap S\) and \(U_\lambda \cap S\) are open sets in \(S\), and \(C_\lambda \cap S\) is closed in \(S\). The family \(\{U_\lambda \cap S\}_{\lambda \in \Lambda}\) is an open cover of \(S\). We will show that it is a regular cover of the subspace \(S\). For each \(\lambda \in \Lambda\), we have that \(\text{cls}\left(\hat{C}_\lambda \cap S\right) \subseteq C_\lambda \cap S \subseteq U_\lambda \cap S\), where \(\text{cls}\left(\hat{C}_\lambda \cap S\right)\) is regularly closed in \(S\). Moreover, we have \(S = \bigcap_{\lambda \in \Lambda} \left( \hat{C}_\lambda \cap S \right) \) and \(\hat{C}_\lambda \cap S \subseteq \bigcup_{\lambda \in \Lambda} \text{cls}\left(\hat{C}_\lambda \cap S\right)\), thus \(S = \bigcup_{\lambda \in \Lambda} \text{cls}\left(\hat{C}_\lambda \cap S\right)\). Since \(S\) is an almost regular-Lindelöf subspace of \(X\), there exists a countable subcover such that \(S = \bigcup_{n \in \mathbb{N}} \text{cls}\left(U_{\lambda_n} \cap S\right)\) if for each \(n \in \mathbb{N}\) \(\text{cls}\left(U_{\lambda_n} \cap S\right) \subseteq U_{\lambda_n}\), we obtain that \(S \subseteq \bigcup_{n \in \mathbb{N}} U_{\lambda_n}\). This shows that \(S\) is almost regular-Lindelöf relative to \(X\).

**THEOREM 4.13.** If every regularly closed subset of a space \(X\) is almost regular-Lindelöf relative to \(X\), then \(X\) is almost regular-Lindelöf.

**PROOF.** Let \(\{U_\lambda\}_{\lambda \in \Lambda}\) be a regular cover of \(X\). For each \(\lambda \in \Lambda\), there exists a nonempty regularly closed set \(C_\lambda\) of \(X\) such that \(C_\lambda \subseteq U_\lambda\) and \(X = \bigcup_{\lambda \in \Lambda} \hat{C}_\lambda\). Fix an arbitrary \(\lambda_0 \in \Lambda\) and let \(\Lambda^* = \Lambda \setminus \{\lambda_0\}\). Put \(K = X \setminus \hat{C}_{\lambda_0}\), then \(K\) is regularly closed in \(X\) and \(K \subseteq \bigcup_{\lambda \in \Lambda^*} \hat{C}_\lambda\). Therefore \(\{U_\lambda\}_{\lambda \in \Lambda^*}\) is a cover of \(K\) by open sets of \(X\) satisfying the condition \(\left( \ast \right)\) of Definition 4.11 and hence there exists a countable subset \(\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \Lambda^*\) such that \(K \subseteq \bigcup_{n \in \mathbb{N}} U_{\lambda_n}\). So we have
\[ X = K \cup \hat{C}_{\lambda_0} = K \cup \bigcup_{n \in \mathbb{N}} U_{\lambda_n} = \left( \bigcup_{n \in \mathbb{N}} U_{\lambda_n} \right) \cup \hat{C}_{\lambda_0} = \bigcup_{n \in \mathbb{N}} U_{\lambda_n} \cup \hat{C}_{\lambda_0}. \]
This shows that \(X\) is almost regular-Lindelöf.

**COROLLARY 4.14.** If every proper regularly closed subset of a space \(X\) is almost regular-Lindelöf, then \(X\) is almost regular-Lindelöf.

**THEOREM 4.15.** Let \(X\) be an almost regular-Lindelöf space. If \(A\) is a proper clopen subset of \(X\), then \(A\) is almost regular-Lindelöf relative to \(X\).
PROOF. Let \( \{U_{\lambda}\}_{\lambda \in \Lambda} \) be a cover of \( A \) by open sets of \( X \) satisfying the condition (\( \ast \)) of Definition 4.11. The family \( \{U_{\lambda}\}_{\lambda \in \Lambda} \cup (X \setminus A) \) is a regular cover of \( X \). Since \( X \) is almost regular-Lindelöf, there exists a countable subfamily such that

\[
X = \left( \bigcup_{\lambda \in N} U_{\lambda} \right) \cup (X \setminus A) = \left( \bigcup_{\lambda \in N} U_{\lambda} \right) \cup (X \setminus A).
\]

Therefore we have \( A \subset \bigcup_{\lambda \in N} U_{\lambda} \). This completes the proof \( \square \).

We conclude this paper introducing the following two definitions:

**DEFINITION 4.16.** A space \( X \) is called weakly regular-Lindelöf if every regular cover \( \{U_{\lambda}\}_{\lambda \in \Lambda} \) of \( X \) admits a countable subfamily such that \( X = \bigcup_{\lambda \in N} U_{\lambda} \).

**DEFINITION 4.17.** A space \( X \) is called nearly regular-Lindelöf if every regular cover \( \{U_{\lambda}\}_{\lambda \in \Lambda} \) of \( X \) admits a countable subfamily such that \( X = \bigcup_{\lambda \in N} U_{\lambda} \).

Obviously we have the following implications

- Nearly-L \( \Rightarrow \) Almost-L \( \Rightarrow \) Weakly-L
- \( \Downarrow \) \( \Downarrow \) \( \Downarrow \)

- Nearly regular-L \( \Rightarrow \) Almost regular-L \( \Rightarrow \) Weakly regular-L

We leave open the study of these two new generalizations of Lindelöf property and the relative implications.

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