ON MATRIX CONVEXITY OF THE MOORE-PENROSE INVERSE

B. MOND
Department of Mathematics
La Trobe University
Bundoora, Victoria, 3083, AUSTRALIA

J.E. PEČARIĆ
Faculty of Textil Technology
University of Zagreb
Zagreb, CROATIA

(Received December 12, 1994 and in revised form June 28, 1995)

ABSTRACT. Matrix convexity of the Moore-Penrose inverse was considered in the recent literature. Here we give some converse inequalities as well as further generalizations.

KEY WORDS AND PHRASES: Matrix convexity, generalized inverse


1. INTRODUCTION
Let $A$ and $B$ be two complex Hermitian positive definite matrices, and let $0 \leq \lambda \leq 1$. Then

$$\lambda A + (1 - \lambda)B \preceq \lambda A^{-1} + (1 - \lambda)B^{-1}$$

(1.1)

where $A \preceq B$ means that $A - B$ is a positive semi-definite matrix.

This result, i.e., matrix convexity of the inverse function is an old result that appears explicitly in the papers [1,2,3,4,5] (see also the books [6, pp. 554-555] and [7, pp. 469-471]).

The related matrix convexity of the Moore-Penrose (generalized) inverse, denoted by $A^+$, was considered in paper [8,9,10]. The following was given in [10]:

Let $A$ and $B$ be two complex Hermitian positive semi-definite matrices of the same order. The inequality

$$\lambda A + (1 - \lambda)B A^+ \preceq \lambda A^+ + (1 - \lambda)B A^+$$

(1.2)

for every $0 \leq \lambda \leq 1$ holds if and only if

$$R(A) = R(B)$$

(1.3)

where $R(A)$ is the range of $A$.

Two converses of (1.1) were obtained in [11]:

If $A$ and $B$ are complex Hermitian positive definite matrices and $0 \leq \lambda \leq 1$ is a real number, then

$$\lambda A + (1 - \lambda)B \preceq \lambda A^{-1} + (1 - \lambda)B^{-1}$$

(1.4)

and

$$\lambda A + (1 - \lambda)B \preceq \lambda A^{-1} + (1 - \lambda)B^{-1} \geq \bar{K} A^{-1}$$

(1.5)

where

$$\bar{K} = 4 \min \frac{\mu_i}{(1 + \mu_i)^2}, \quad \bar{K} = \min \frac{\sqrt{\mu_i} - 1}{-\mu_i},$$

(1.6a,b)

and the $\mu_i$ are the solutions of the equation

$$\det(B - \mu A) = 0.$$  

(1.7)

In this note, we give analogous converses for (1.2), as well as some related results.

2. CONVERSES OF THE MATRIX CONVEXITY INEQUALITY OF THE MOORE-PENROSE INVERSE
Let $A$ and $B$ be two complex Hermitian positive semi-definite matrices of the same order such that (1.3) holds. Let $P$ be a unitary matrix such that $A = P \operatorname{diag}(A_1, 0)P^*$ and $B_1$ is a diagonal positive definite matrix. When (1.3) holds, we have $B = P \operatorname{diag}(B_1, 0)P^*$ where $B_1$ is positive definite.

In this note, we give analogous converses for (1.2), as well as some related results.
**THEOREM 1.** Let $A$ and $B$ be two complex Hermitian positive semi-definite matrices of the same order such that (1 3) holds and let $0 \leq \lambda \leq 1$ Then

$$[\lambda A + (1 - \lambda)B]^* \geq K(\lambda A^* + (1 - \lambda)B^*)$$  \hspace{1cm} (2.1)

where $K$ is defined by (1 6a) and the $\mu_i$ are the positive solutions of the equation

$$\text{det}(B_1 - \lambda A_1) = 0.$$  \hspace{1cm} (2.2)

**THEOREM 2.** Let $A, B$ be defined as in Theorem 1. Then

$$[\lambda A + (1 - \lambda)B^*] - (\lambda A^* + (1 - \lambda)B^*) \geq K A^*$$  \hspace{1cm} (2.3)

where $K$ is defined by (1 6b) and the $\mu_i$ are positive solutions of the equation (2.2)

**PROOF.** By (1 4) and (1 5) we have

$$[\lambda A_1 + (1 - \lambda)B_1]^{-1} \geq K(\lambda A_1^{-1} + (1 - \lambda)B_1^{-1})$$  \hspace{1cm} (2.4)

and

$$[\lambda A_1 + (1 - \lambda)B_1]^{-1} - (\lambda A_1^{-1} + (1 - \lambda)B_1^{-1}) \geq K A_1^{-1}$$  \hspace{1cm} (2.5)

where $K$ is defined by (1 6a), $K$ by (1 6b) and the $\mu_i$ are solutions of (2.2). Since $PA^*P^* = (PA^*)^*$, (2.1) follows from (2.4) and (2.3) from (2.5).

3. SOME RELATED RESULTS

Let $(Y, B, \mu)$ be a probability space and $A_y, y \in Y$ a collection of positive semi-definite matrices of the same order. Let $A_y = (a_{ijy}), 1 \leq i, j \leq n$ and $y \in Y$. Assume that $a_{ijy}$ as a function of $y$ is measurable for every $1 \leq i, j \leq n$. The following results were proved in [9, 10].

Suppose there exists a set $D \in B$ such that $\mu(D) = 1$ and $A_{y1}A_{y2} = A_{y2}A_{y1}$ for every $y_1, y_2 \in D$. Let $R(A_y)$ be the same for all $y \in D \in B$. Suppose $A_y$ and $A_y^+$ as functions of $y$ are integrable with respect to $\mu$ Then

$$\left(\int_Y A_y \mu(dy)\right)^+ \leq \int_Y A_y^+ \mu(dy).$$  \hspace{1cm} (3 1)

By $\int_Y A_y \mu(dy)$ we mean the matrix whose $(i, j)^{th}$ element is $\int_Y a_{ijy} \mu(dy)$.

**THEOREM 3.** If also all positive eigenvalues of $A_y$ for all $y \in Y$ are in the interval $[m, M]$ where $0 < m < M$, then the following inequalities hold:

$$\int_Y A_y^+ \mu(dy) \leq \frac{(M + m)^2}{4Mm} \left[\int_Y A_y \mu(dy)\right]^+$$  \hspace{1cm} (3 2)

and

$$\int_Y A_y^+ \mu(dy) - \left[\int_Y A_y \mu(dy)\right]^+ \leq \frac{(\sqrt{M} - \sqrt{m})^2}{Mm} I.$$  \hspace{1cm} (3 3)

**PROOF.** As in [9], we have that there exists an orthogonal matrix $C$ such that

$C^TAC = \text{diag}\{\lambda_{1y}, \lambda_{2y}, ..., \lambda_{ny}\}, \ y \in Y$ 

where $\lambda_{1y}, \lambda_{2y}, ..., \lambda_{ny}$ are the eigenvalues of $A_y$. Since $A_y$ is positive semi-definite, each $\lambda_{iy} \geq 0$. Let $k$ be the rank of $A_y$. We can assume without loss of generality that

$\lambda_{1y}, \lambda_{2y}, ..., \lambda_{ky} \neq 0$ for every $y \in Y$, and $\lambda_{k+1,y} = \lambda_{k+2,y} = ... \lambda_{ny} = 0$ for every $y \in Y$

Note that

$A_y^+ = C \text{diag}\left\{\frac{1}{\lambda_{1y}}, \frac{1}{\lambda_{2y}}, ..., \frac{1}{\lambda_{ky}}, 0, ..., 0\right\}C^T$

so that
Thus, we have
\[
K \left[ \int Y A_\mu (dy) \right]^{-1} - \int Y \lambda_{1y}^{-1} \mu (dy) = C \text{ diag} \left\{ K \left( \int Y \lambda_{1y} \mu (dy) \right)^{-1}, \ldots, K \left( \int Y \lambda_{ky} \mu (dy) \right)^{-1} - \int Y \lambda_{ky}^{-1} \mu (dy), 0, \ldots, 0 \right\} \in \mathbb{R}^n
\]

where \( K = (M + m)^2 / (4Mm) \) The inequality
\[
K \left[ \int Y \lambda_{1y} \mu (dy) \right]^{-1} \int Y \lambda_{1y}^{-1} \mu (dy)
\]

is the well-known Kantorovich inequality Hence each diagonal element in the above diagonal matrix is non-negative This completes the proof of (3.2)

Similarly,
\[
\int Y A_\mu^{-1} (dy) = \int Y A_\mu (dy) \left[ \int Y A_\mu (dy)^{-1} - K I \right] = C \text{ diag} \left\{ \int Y \lambda_{1y}^{-1} \mu (dy) - \left( \int Y \lambda_{1y} \mu (dy) \right)^{-1}, \ldots, \int Y \lambda_{ky}^{-1} \mu (dy) - \left( \int Y \lambda_{ky} \mu (dy) \right)^{-1} - K, - K, \ldots, - K \right\} \in \mathbb{R}^n
\]

where \( \bar{K} = \frac{(\sqrt{M} - \sqrt{m})^2}{Mm} \) The inequality
\[
\int Y \lambda_{1y}^{-1} \mu (dy) - \int Y \lambda_{1y} \mu (dy)^{-1} \leq \bar{K}
\]

is a simple consequence of the following Mond-Shisha inequality [12]
\[
\int f - \left( \int f^{-1} \right)^{-1} \leq (\sqrt{M} - \sqrt{m})^2
\]

where \( m \leq f \leq M, 0 < m < M \). Namely
\[
\frac{1}{M} \leq \frac{1}{f} \leq \frac{1}{m} \quad \text{so that by substituting } f \to \frac{1}{f}, \text{ we get}
\]
\[
\int f^{-1} - \left( \int f \right)^{-1} \leq \frac{(\sqrt{M} - \sqrt{m})^2}{Mm} = \bar{K}
\]

Thus each diagonal element in the above diagonal matrix is non-positive. This completes the proof

Moreover, we can consider the powers of \( A \) and \( A^+ \). For simplicity of notation, if \( r < 0 \), we shall use \( A^{(r)} \) for \( (A^+)^{-r} \). Note that \( (A^+)^{-1} = (A^{-1})^+ \)

**Theorem 4.** Let \( R(A_\mu) \) be the same for all \( y \in \mathcal{D} \in \mathcal{B} \). Suppose \( A_{\mu}^r \) and \( A_{\mu}^{(r)} \) \((r < 0 < s)\) as functions of \( y \) are integrable with respect to \( \mu \) Then
\[
\left[ \int Y A_{\mu}^{(r)} (dy) \right]^s \geq \left[ \int Y A_{\mu}^r (dy) \right]^{(r)} \quad (3.4)
\]

**Proof.** As in the proof of (3.2) and (3.3), we have
\[
\left[ \int Y A_{\mu}^{(r)} (dy) \right]^s - \left[ \int Y A_{\mu}^r (dy) \right]^{(r)} = C \text{ diag} \left\{ \left( \int Y \lambda_{1y}^r \mu (dy) \right)^s - \left( \int Y \lambda_{1y}^r \mu (dy) \right)^{r}, \ldots, \right. \]
\[
\left. \left( \int Y \lambda_{ky}^r \mu (dy) \right)^s - \left( \int Y \lambda_{ky}^r \mu (dy) \right)^{r}, 0, \ldots, 0 \right\} \in \mathbb{R}^n
\]
Each diagonal element in the above diagonal matrix is nonnegative. This follows from the fact that if $f^s$ and $f^r$ are positive and integrable, the well-known inequality for means of orders $s$ and $r$ states that

$$\left( \int f^r \right)^{1/r} \leq \left( \int f^s \right)^{1/s} \quad (r < 0 < s)$$

which is the same as

$$\left( \int f^s \right)^r \leq \left( \int f^r \right)^s.$$

Similar consequences of converse inequalities for (3.5) (see [12] and [13], respectively) are the next two theorems.

**THEOREM 5.** Let the conditions of Theorem 4 be satisfied and let all positive eigenvalues of $A_y$ for all $y \in Y$ belong to the interval $[m, M]$ ($0 < m < M$). Then the following inequality holds

$$\left[ \int_Y A_y^s \mu(dy) \right]^{(r)} \geq \Delta \left[ \int_Y A_y^{(r)} \mu(dy) \right]^s \tag{3.6}$$

where

$$\Delta = \left\{ \begin{array}{c} r(\gamma^s - \gamma^r) \\ (s - r)(\gamma^r - 1) \end{array} \right\}^{-1} \left\{ \begin{array}{c} s(\gamma^r - \gamma^s) \\ (r - s)(\gamma^s - 1) \end{array} \right\}^{-s}, \quad \gamma = M/m. \tag{3.7}$$

**THEOREM 6.** Let the conditions of Theorem 5 be satisfied. Then

$$\left[ \int_Y A_y^{(r)} \mu(dy) \right]^s - \left[ \int_Y A_y^s \mu(dy) \right]^{(r)} \leq \Lambda I \tag{3.8}$$

where

$$\Lambda = \max_{\theta \in [0,1]} \{[\theta M^r + (1 - \theta)m^r]^s - [\theta M^s + (1 - \theta)m^s]^r\}.$$

Of course (3.2) and (3.3) are the special cases $r = -1, s = 1$ of (3.6) and (3.8).

**REFERENCES**


