RESEARCH NOTES

A CURIOUS INTEGRAL

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(Received November 22, 1994)

ABSTRACT. A double integral which came from a cohomology calculation is evaluated explicitly using the properties of \( \,_3F_2 \) and \( \,_2F_1 \) hypergeometric functions.

KEY WORDS AND PHRASES: double integral, hypergeometric functions.

1992 AMS SUBJECT CLASSIFICATION CODE: 33C90.

1. INTRODUCTION.

The problem of evaluating the integral

\[
\int_0^{\pi/2} \int_0^{\pi/2} \frac{(1 - 4 \cos^2 s \cos^2 t)}{(1 + 8 \cos^2 s \cos^2 t)^{3/2}} \, ds \, dt
\]

has been proposed by A. Lundell. The computer algebra language Maple tells the user that it cannot be evaluated explicitly but evaluates it numerically to seven decimal places in a couple of seconds. Mathematica, on the other hand, reduces it to the evaluation of a single integral by performing one of the single integrals.

The integral arose as a reduction of a surface integral on a torus which came in relating the cohomology of \( \mathbb{R}^3 - (C \cup L) \) and \( \mathbb{R}^3 - C \) where \( C \) is the circle \( x^2 + y^2 = a^2 \) in the \( xy \)-plane and \( L \) is the \( z \)-axis and where numerical calculations suggested the value \( \pi/4 \) [2, p.19]. The purpose of this note is to prove this conjecture.

We first consider the more general integral

\[
I(a, b, c) := \int_0^{\pi/2} \int_0^{\pi/2} \frac{(1 + b \cos^2 s \cos^2 t)}{(1 + a \cos^2 s \cos^2 t)^{c/2}} \, ds \, dt. \tag{1.1}
\]

We find that \( I(a, b, c) \) can be expressed as a sum of two \( \,_3F_2 \)'s with argument \(-a\). Although there are no explicit general formulas for the analytic continuation of \( \,_3F_2 \)'s something remarkable happens when \( c = 3/2 \). In this case each \( \,_3F_2 \) can be expressed as a product of \( \,_2F_1 \)'s of argument \(-a\) which may now be analytically continued throughout the complex \( a \)-plane cut along \((\infty, -1]\). A further simplification occurs when \( b = -4 \) with \( I(a, -4, 3/2) \) being expressed as a single product of two \( \,_2F_1 \)'s. A final remarkable simplification occurs with \( a = 8 \)
when each of these $\genfrac{[}{]}{0pt}{}{r+1}{1}$'s can be explicitly summed in terms of gamma functions. As an end result we then obtain

**THEOREM 1.**

$$I(8, -4, 3/2) = \frac{\pi}{4}. \quad (1.2)$$

In the next section we prove this result using the theory of hypergeometric functions where

$$r+1 F_r \left( \begin{array}{c} a_1, a_2, \cdots, a_r+1 \nonumber \n \end{array} \begin{array}{c} b_1, b_2, \cdots, b_r \nonumber \n \end{array} ; z \right) := \sum_{n=0}^{\infty} \frac{\left( a_1, a_2, \cdots, a_r+1 \right)_n}{\left( b_1, b_2, \cdots, b_r \right)_n} \frac{z^n}{n!}, \quad (1.3)$$

$$(a)_n = \Gamma(a+n)/\Gamma(a), \quad (a_1, a_2, \cdots, a_r)_n = \prod_{j=1}^{r} (a_j)_n.$$ 

The following formulas will be needed.

$$\genfrac{[}{]}{0pt}{}{2a+2b-2, a+\beta-1}{2a+2\beta-2, a+\beta-1/2} \cdot \frac{\Gamma(a+b+1)}{\Gamma(a+\beta+1/2)} = (1-z)^{-a} \cdot \frac{\Gamma(c-a-b)}{\Gamma(1-a-b)} \cdot \frac{\Gamma(c-a)}{\Gamma(1-a)}, \quad (1.6)$$

$$\genfrac{[}{]}{0pt}{}{a+b}{c} \cdot (1-z)^{-a} \cdot \frac{\Gamma(c-a-b)}{\Gamma(1-a-b)} \cdot \frac{\Gamma(c-a)}{\Gamma(1-a)} = 0, \quad (1.8)$$

$$\genfrac{[}{]}{0pt}{}{a+b}{c+1} = \frac{\Gamma(1+a+b)}{\Gamma(1+b+a/2)} \cdot \frac{\Gamma(1/2+a)}{\Gamma(1/2+a/2)} \cdot \frac{\Gamma(1/2+a/2)}{\Gamma(1-b+a/2)} \cdot \frac{\Gamma(1-a)}{\Gamma(1-a)}, \quad (1.10)$$

$$\genfrac{[}{]}{0pt}{}{a+b}{1+a-b} = 2^{-a} \cdot \frac{\Gamma(1+a-b)}{\Gamma(1-b+a/2)} \cdot \frac{\Gamma(1/2+a/2)}{\Gamma(1/2+a/2)}. \quad (1.11)$$

These formulas are in [1], (9) and (8) p. 186, (3) and (2) p. 105, (30) p. 103, (10) and (13) p. 111, and (47) p. 104 respectively.

2. **THE PROOF.**

To prove Theorem 1 we first establish four lemmas.

**LEMMA 2.1.** Let

$$u_n := \int_{0}^{\pi/2} \cos^{2n} t \, dt, \quad n = 0, 1, \cdots. \quad (2.1)$$

Then

$$u_n = \frac{\pi (1/2)^n}{n!}. \quad (2.2)$$

PROOF. This result is well known. An integration by parts yields \( u_n = \frac{2n-1}{2n} u_{n-1}, n \geq 1 \).

Clearly \( u_0 = \pi/2 \). Iterating we get (2.2).

**LEMMA 2.2.** If \( |a| < 1 \) then

\[
I(a, b, c) = \frac{\pi^2}{4} \left[ \, _3F_2 \left( \frac{5}{4}, \frac{1}{2}, c; \frac{1}{4}, 1 ; -a \right) + \frac{b}{4} \, _3F_2 \left( \frac{3}{4}, \frac{3}{2}, c; \frac{1}{4}, 1 ; -a \right) \right].
\]

**PROOF.** In (1.1) we expand \((1 + a \cos^2 x \cos^2 t)^{-c}\) using the binomial theorem and do the integration. Using Lemma 2.1 we then obtain (2.3).

We now specialize to the value \( c = 3/2 \).

**LEMMA 2.3.** If \( |a| < 1 \) or \( a = 1 \) then

\[
I(a, b, 3/2) = \frac{\pi^2}{4} \left[ \, _3F_2 \left( \frac{5}{4}, \frac{1}{2}, 1; \frac{1}{4}, 1 ; -a \right) + \frac{b}{4} \, _3F_2 \left( \frac{3}{4}, \frac{3}{2}, 1; \frac{1}{4}, 1 ; -a \right) \right].
\]

**PROOF.** We use (1.4) for the first \(_3F_2\) on the right of (2.3) and (1.5) for the second \(_3F_2\) on the right of (2.3).

Having established (2.4) for \( |a| < 1 \) one may use the properties of \(_2F_1\)’s to obtain an analytic continuation of (2.4) throughout the complex \( a \)-plane cut along \((-\infty, -1]\).

We now specialize to the values \( b = -4, c = 3/2 \).

**LEMMA 2.4.**

\[
I(a, -4, 3/2) = \frac{15\pi^2 a}{128} \left[ \, _2F_1 \left( \frac{1}{4}, \frac{1}{2}; \frac{1}{4}, 1 ; -a \right) + \frac{b}{4} \, _2F_1 \left( \frac{3}{4}, \frac{3}{2}; \frac{1}{4}, 1 ; -a \right) \right].
\]

**PROOF.** In (2.4) we put \( b = -4 \) and apply (1.7) to the first and third \(_2F_1\) on the right of (2.4). The result is

\[
I(a, -4, 3/2) = \frac{\pi^2}{4(1 + a)^{1/2}} \left[ \, _2F_1 \left( \frac{1}{4}, \frac{1}{2}; \frac{1}{4}, 1 ; -a \right) + \frac{b}{4} \, _2F_1 \left( \frac{3}{4}, \frac{3}{2}; \frac{1}{4}, 1 ; -a \right) \right].
\]

We now apply (1.6) to the \(_2F_1\)’s in the brackets above and then use (1.8). This gives

\[
I(a, -4, 3/2) = \frac{15\pi^2 a}{128(1 + a)^{5/4}} \, _2F_1 \left( \frac{1}{4}, \frac{1}{2}; \frac{1}{4}, 1 ; -a \right) \, _2F_1 \left( \frac{3}{4}, \frac{3}{2}; \frac{a}{1 + a} \right).
\]

After another application of (1.6) to the second \(_2F_1\) above we obtain (2.5).

**PROOF OF THEOREM 1.** We now specialize to the case \( a = 8, b = -4, c = 3/2 \). In (2.5) we put \( a = 8 \). We use (1.9) and (1.11) to get

\[
_2F_1 \left( \frac{1}{4}, \frac{1}{2}; \frac{1}{4}, 1 ; -8 \right) = \frac{\Gamma(1)\Gamma(1/2)}{\Gamma(3/4)^2}.
\]

Using (1.10) and (1.11) we also get

\[
_2F_1 \left( \frac{5}{4}, \frac{9}{4}; \frac{1}{4}, 1 ; -8 \right) = \frac{\Gamma(3)\Gamma(1/2)}{3\Gamma(5/4)\Gamma(9/4)^2}.\]
Thus

\[ I(8, -4, 3/2) = \frac{\pi^2 \Gamma^2(1/2)}{32 \Gamma^2(3/4) \Gamma^2(5/4)} \quad (2.10) \]

where we have used the above $2\!F_1$ evaluations together with $\Gamma(1) = 1, \Gamma(3) = 2$ and $\Gamma(9/4) = 5\Gamma(5/4)/4$. A final use of the duplication formula [1.(15), p. 5] yields $\Gamma^2(1/2) = \pi, \Gamma^2(3/4) \Gamma^2(3/4) = \pi^2/8$ and the theorem is established.

ACKNOWLEDGEMENT. This research was partially supported by NSERC (Canada).

REFERENCES