EXOTIC STRUCTURES ON QUOTIENT SPACES OF $S^3$-ACTIONS

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(Received May 4, 1994)

ABSTRACT. A correct version of some results by A. Rigas regarding $S^3$ actions on $S^7 \times S^3$ and on the symplectic group $Sp_2$ with quotients exotic seven-spheres is presented

KEY WORDS AND PHRASES: Exotic spheres, principal bundles, group actions

1991 AMS SUBJECT CLASSIFICATION CODES: 57R55, 57S25

1. INTRODUCTION

The present note is a result of our interest in finding exotic structures on 7-dimensional manifolds (cf Guest and Micha [3], Astey, Micha and Pastor [1]) and its purpose is to correct some mistakes that occur in a paper by A. Rigas [6]. Our contribution is simply to provide the correct statement and a different proof of the key corollary that appears on page 76 of Rigas [6], but we take the opportunity to restate several results of the paper which refer to the existence of free $S^3$ actions on $S^7 \times S^3$ and on the symplectic group $Sp_2$ with quotients exotic seven-spheres, which also appear incorrectly stated in that paper.

2. MAIN RESULTS

We begin by recalling some definitions and notation of Rigas [6]. Principal $S^3$ bundles over $S^4$ are classified by $\pi_3 S^3$ which is naturally isomorphic to the group of integers $\mathbb{Z}$. Let $P_n$ denote the total space of the bundle corresponding to the integer $n$. Similarly, the principal $S^3$ bundles over $S^7$ are classified by $\pi_6 S^3$. We shall denote by $E_i$ the total space of the bundle corresponding to $i \in \pi_6 S^3 \cong \mathbb{Z}_{12}$. The isomorphism here is such that $E_1 \cong Sp_2$. Let $\tilde{P}_n$ denote the pull-back of $P_n$ under the Hopf map $S^7 \to S^4$. Then, as a principal $S^3$ bundle, $\tilde{P}_n$ is classified by the composition

$$S^7 \xrightarrow{h} S^4 \xrightarrow{f_n} S^4 \to BS^3$$

where $f_n$ denotes the map of degree $n$, and the rightmost arrow is the inclusion of the bottom cell.

THEOREM. The bundles $\tilde{P}_n$ and $E_{(n+1)/2}$ are isomorphic as principal $S^3$ bundles over $S^7$.

This theorem is the correct version of the corollary on page 76 of Rigas [6]. The mistake leading to the incorrect statement in Rigas [6] occurs in the calculation of the map $f_n \circ h$, where the author fails to
iterate correctly a formula of Hilton [4]. An alternative proof using a different bundle decomposition is presented in §3 below.

It follows from the theorem that
(a) $\hat{P}_n$ and the trivial bundle $S^7 \times S^4$ are isomorphic only if $n \equiv 0, 1, 9$ or $16$ mod $24$
(b) $\hat{P}_n$ and the canonical bundle $Sp_2 \to S^7$ are isomorphic only if $n \equiv 2$ or $23$ mod $24$

In particular, $\hat{P}_{14}$ is not a trivial bundle. This renders §4 of Rigas [6] invalid. The theorem also allows us to rectify the statements of two important results of Rigas [6] as follows

**COROLLARY.** There exist free actions of $S^3$ on $S^7 \times S^4$ with quotient the exotic seven-spheres of Eells-Kuiper invariants $16, 40$ and $48$

**COROLLARY.** There exist free actions of $S^3$ on $Sp_2$ with quotient the exotic seven-spheres of Eells-Kuiper invariants $2, 26, 34$ and $42$

3. PROOF OF THE THEOREM

As is shown in Rigas [6], $S^7$ can be decomposed into two solid tori $U \cong S^1 \times D^4$ and $V \cong D^4 \times S^1$ such that the restriction of the bundle $\hat{P}_n$ to each torus is trivial. Moreover, the transition map

$$\lambda_{UV} : S^3 \times S^3 \to S^3$$

is given by

$$\lambda_{UV}(x, y) = x^{n-1}(yx^{-1})^{n-1}y^{n-1},$$

where the group structure of unit quaternions is understood on $S^3$. Since the commutator $xyz^{-1}y^{-1}$ generates $\pi_6S^3$ (Hilton and Roitberg [5]) and since $\lambda$ factors through $S^6$, the theorem is a consequence of the following result

**PROPOSITION.** The map $\lambda : S^3 \times S^3 \to S^3$ given by $\lambda(x, y) = x^{n-1}(yx^{-1})^{n-1}y^{n-1}$ is homotopic to $(xyz^{-1}y^{-1})^{n(n-1)/2}$

We first prove the following lemma

**LEMMA.** The maps $x^ky/y^{-1}$ and $(xyz^{-1}y^{-1})^{kl}$ are homotopic.

**PROOF.** Consider the following commutative diagram

$$\begin{array}{ccc}
S^3 \times S^3 & \xrightarrow{\alpha} & S^3 \\
\downarrow p & & \downarrow p \\
S^6 & \xrightarrow{\omega} & S^6 \\
\end{array}$$

where $\alpha(x, y) = (x^k, y^l)$, $\beta(x, y) = x^ky^{-1}y^{-1}$, $\gamma(x) = x^{kl}$, $p$ is the projection that collapses the 3-skeleton, $f_{kl}$ is a map of degree $kl$, and $\omega$ is the generator of $\pi_6S^3$. But since $S^3$ is an H-space, homotopy compositions are biadditive (Whitehead [7], p. 479), so $\omega \circ f_{kl} \simeq \gamma \circ \omega$. Therefore,

$$x^ky^lx^{-k}y^{-l} = \beta \circ \alpha \simeq \gamma \circ \beta = (xyz^{-1}y^{-1})^{kl}$$

We now prove the proposition by induction on $n$. Let $c = x^{y^{-1}}y^{-1}$. If we take $k = 1$ and $l = -1$ in the lemma we obtain $xy^{-1}x^{-1}y^{-1} \simeq c^{-1} = yxy^{-1}x^{-1}$. Hence,

$$c^{-1}y_c^{-1} = (yxy^{-1}x^{-1})y_c^{-1} = y(xy^{-1}x^{-1}y)c^{-1} \simeq yc^{-1}cy^{-1} = 1,$$

that is, $cy^{-1} = y^{-1}c$.

Assume now that $x^n(yx^{-1})^ny^{-n} = c^{k(n)}$. Clearly, $k(1) = 1$. But now
\[ x^n(yx^{-1})y^{-n} = x^n y x^{-1} (yx^{-1})^{n-1} y^{-n} \]
\[ = (x^n y x^{-1} y^{-1}) y (x^{-1})^{n-1} y^{-n} \]
\[ = c^n y (x^{-1})^{n-1} y^{-(n-1)} y^{-1} \]
\[ = c^n y c^{(n-1)} y^{-1} \]
\[ = c^{n+k(n-1)}. \]

Therefore, \( k(n) = n + k(n - 1) \), that is, \( k(n) = n(n + 1)/2 \) This proves the proposition

REFERENCES


