APPLICATION ON LOCAL DISCRETE EXPANSION

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ABSTRACT. The process of changing a topology by some types of local discrete expansion preserves s-closeness, S-closeness, semi-compactness, semi-T_i, semi-R_i, i {0, 1, 2}, and extremely disconnectedness. Via some other forms of such above replacements one can have topologies which satisfy separation axioms the original topology does not have.

KEY WORDS AND PHRASES: Near open sets, local discrete expansion, extremely disconnected, semi-compact, s-closed, S-closed, semi-T, semi-\( \tau \)-, and cid spaces

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1. INTRODUCTION

Throughout the present paper \((X, \tau)\) is a topological space (or simply a space \(X\)) on which no separation axioms are assumed unless explicitly stated. For any \(B \subseteq X\), \(cl_B\) (resp \(int_B\)) denotes the closure (resp interior) of \(B\). A subset \(B\) is said to be regular open (resp regular closed) if \(B = int(cl_B)\) (resp \(cl_B = int_B\)). A subset \(B\) of a space \(X\) is said to be \(\tau\)-semi open [12] (resp \(\tau\)-regular semi-open [2]) if there exists a \(\tau\)-open (resp \(\tau\)-regular open) set \(U\) satisfying \(U \subseteq B \subseteq cl_U\). \(B\) is \(\tau\)-semi-closed [3] if the set \(X - B\) is \(\tau\)-semi-open. The family of all regular open (resp regular semi-open, semi-open) sets in \(X\) is denoted by \(RO(X, \tau)\) (resp \(RSO(X, \tau), SO(X, \tau)\)). The union (resp intersection) of all \(\tau\)-semi-open (resp \(\tau\)-semi-closed) sets contained in \(B\) (resp containing \(B\)) is called the \(\tau\)-semi-interior (resp \(\tau\)-semi-closure) of \(B\), and it is denoted as \(s-int_B\) (resp \(s-clu_B\)). A space \(X\) is said to be extremely disconnected (denoted by E.D.) if for every open set \(U\) of \(X\), \(cl_U\) is open in \(\tau\) (The concept of local discrete expansion of a topology was first introduced by S.P. Young in 1977 [17], "Let \((X, \tau)\) be a topological space and \(A\) be any subset of \(X\). The topology \(\tau[A] = \{U - H : U \in \tau, H \subseteq A\}\) is called the local discrete expansion of \(\tau\) by \(A\). A space \(X\) is semi-T_2 [13] (resp semi-T_3 [11]) iff for \(x, y \in X, x \neq y\) there exist \(U, V \in SO(X, \tau)\) such that \(cl_U \cap cl_V = \emptyset\) (resp \(cl_U \cup cl_V = \emptyset\)). Semi-T_0 and semi-T_1 were introduced to topological spaces [13] by replacing the word "open" by "semi-open" in the definitions of T_0 and T_1 respectively. A space \(X\) is semi-R_0 [6] iff for each semi-open set \(U\) and \(x \in U\), \(s-cl_U\{x\} \subseteq U\). A space \(X\) is semi-R_1 [6] iff for \(x, y \in X\) such that \(s-cl_U\{x\} \neq s-cl_U\{y\}\) there exist disjoint semi-open sets \(U, V\) such that \(s-cl_U\{x\} \subseteq U\), and \(s-cl_U\{y\} \subseteq V\). A space \(X\) is called cid [15] if every countable infinite subspace of \(X\) is discrete. A space \(X\) is semi-compact [7] (resp s-closed [5], S-closed [16]) if for every cover \(\{U_i : i \in I\}\) of \(X\) by semi-open sets of \(X\), there exists a finite subset \(I_0\) of \(I\) such that \(X = \cup \{U_i : i \in I_0\}\) (resp \(X = \cup slc(V_i) : i \in I_0\), \(X = \cup cl(V_i) : i \in I_0\)).

REMARK 1.1. For a subset \(A\) of a space \((X, \tau)\) we say that \(A\) satisfies condition \((C_1)\) if \(A \cup U = \emptyset\), for every \(U \in \tau - \{X\}\).

Listed below are theorems that will be utilized in this paper:

THEOREM 1.1 [14] If \(\tau\) and \(\tau'\) are two topologies on \(X\) such that \(\tau \subset \tau'\), then \(RO(X, \tau) = RO(X, \tau')\) iff \(cl_G = cl_{\tau'}G\) for every \(G \in \tau'\) [equivalent iff \(int_F = int_{\tau'}F\), for every \(F \in \tau'^{\tau'}\)]

THEOREM 1.2 [11] If \(X\) is a space, and \(A \subseteq X\) satisfying \((C_1)\). Then, \(cl_{\tau[A]}G = cl_G\), for every \(G \in \tau[A]\).
THEOREM 1.3 [4] If $X$ is a space, and $A \in SO(X, \tau)$ such that $A \subset B \subset cl_A A$ Then, $B \in SO(X, \tau)$

THEOREM 1.4 [10] If $X$ is a space, and $B \subset X$, then $s - cl_B B = B \cup int, cl_B B$

THEOREM 1.5 [8] A space $X$ is E D iff for every pair $U$ and $V$ of disjoint $\tau$-open sets, we have $cl_U \cap cl_V = \phi$

THEOREM 1.6 [5] A space $X$ is $s$-closed iff every cover of $X$ by regular semi-open sets has a finite subcover

THEOREM 1.7 [15] (a) A space $X$ is cid if every countable infinite subset is closed
(b) Any infinite cid space is $T_1$

THEOREM 1.8 [17] Let $A$ be any subset of $X$ Then $(A, \tau[A] \cap A)$ is discrete

THEOREM 1.9 [17] Let $A$ be a closed subset of $X$ Then $(A, \tau\cap A)$ is a discrete subspace of $X$ iff $\tau = \tau[A]$

THEOREM 1.10 [9] Let $X$ be a $T_1$-space Then $X$ is cid iff countable subsets have no limits points

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THEOREM 2.1. If $(X, \tau)$ is a space and $A \subset X$, then
(i) $SO(X, \tau[A]) \subset \{B - H : B \in SO(X, \tau), H \subset A\}$
(ii) If $A$ satisfying $(C_1)$, then the inclusion symbol in (i) is replaced by equality sign

PROOF. (i) Let $W \in SO(X, \tau[A])$, then there exists $V \in \tau[A]$ such that $V \subset W \subset cl_{\tau[A]} V$
Then $(U - H_1) \subset W \subset cl_{\tau[A]}(U - H_1)$, where $U \in \tau, H_1 \subset A$ 
Put $H_2 = U \cap H_1$, then $H_2 \subset A$, and $(U - H_1) \cup H_2 \subset W \subset cl_{\tau[A]}(U - H_1) \cup H_2$ Then $U \subset W \cup H_2 \subset cl_{\tau[A]} U \subset cl_{\tau[A]} U$
and $(W \cup H_2) \in SO(X, \tau)$ Put $B = W \cup H_2$, and $H = H_1 - W \subset A$ Then $B - H = W \cup (U \cap H_1) - (H_1 - W) = W$.

(ii) By Theorem 1.2, the proof is obvious

REMARK 2.1. From Theorem 2.1, it is easy to prove that, for any $A \subset X$
$SO(X, \tau) \subset SO(X, \tau[A])$

THEOREM 2.2. If $(X, \tau)$ is a space, and $A \subset X$ satisfying $(C_1)$ Then
(i) $SO(X, \tau) = SO(X, \tau[A])$
(ii) $RSO(X, \tau) = RSO(X, \tau[A])$

PROOF. In general $SO(X, \tau) \subset SO(X, \tau[A])$. To prove the converse, let $W \in SO(X, \tau[A])$, then there exists $V \in \tau[A]$ satisfying $V \subset W \subset cl_{\tau[A]} V$. Then $(U - H) \subset W \subset cl_{\tau[A]}(U - H)$, $U \in \tau$, $H \subset A$. There are two cases.
(a) $U \neq X$, then $U - H = U$ Since $cl_{\tau[A]} U = cl_U U$, then $W \in SO(X, \tau)$.
(b) $U = X$, then $(X - H) \subset W \subset cl_{\tau[A]}(X - H) \subset cl_{\tau[A]}(X - H)$. Since $A \cap U = \phi$, then
$cl_{\tau[A] U = \phi}$. Moreover, $cl_{\tau[A]} H \cap U = \phi$, implies to $cl_{\tau[A]} H \cap U = \phi$, for each $U \in \tau - \{X\}$. Hence $U \notin \tau, cl_{\tau[A]} H \cap \int, cl_{\tau[A]} H = \phi$, and $H$ is a $\tau$-semi-closed set. Thus $(X - H) \in SO(X, \tau)$ From Theorem 1.3, $W \in SO(X, \tau)$

(ii) By Theorems 1.1 and 1.2, the proof is obvious

COROLLARY 2.1. If $X$ is a space, and $A \subset X$ satisfying $(C_1)$ Then
(i) $(X, \tau)$ is semi-$T_i$ iff $(X, \tau[A])$ is semi-$T_i$ (i.e. $\{0, 1, 2\}$).
(ii) If $(X, \tau)$ is semi-$T_2$, then $(X, \tau[A])$ is semi-$T_2$
(iii) If $(X, \tau)$ is semi-$R_i$, then $(X, \tau[A])$ is semi-$R_i$ (i.e. $\{0, 1\}$)

PROOF. By Theorem (2.2), the proof is obvious

THEOREM 2.3. If $X$ is a space, and $A \subset X$ satisfying $(C_1)$. Then $s - cl_{\tau[A]} G = s - cl_G$, for every $G \in \tau[A]$

PROOF. Let $G \in \tau[A]$, then $s - cl_{\tau[A]} G = G \cup \int, cl_{\tau[A]} G = G \cup \int, cl_G = s - cl_G$ by Theorems 1.1, 1.2 and 1.4]
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THEOREM 2.4. If \( X \) is a space, and \( A \subset X \) satisfying (C1). Then \( (X, \tau) \) is E.D. iff \( (X, \tau[A]) \) is E.D.

PROOF. Let \( (X, \tau) \) be E.D., \( W \in \tau[A] \) Then \( W = U - H, U \in \tau, H \subset A \).

But \( cl_{\tau[A]}(U - H) = cl_{\tau[A]}U = cl_{\tau}U \), and \( cl_{\tau}U \in \tau \). Thus \( cl_{\tau[A]}W \in \tau[A], \) and \( (X, \tau[A]) \) is E.D.

Conversely, let \( (X, \tau[A]) \) be E.D., and \( U, V \in \tau \) such that \( cl_{\tau}U \cap cl_{\tau}V \neq \emptyset \). By Theorem 1.2, \( cl_{\tau[A]}U \cap cl_{\tau[A]}V \neq \emptyset \), then \( U \cap V \neq \emptyset \) [by Theorem 1.5]. Hence \( (X, \tau) \) is E.D.

THEOREM 2.5. If \( X \) is a space, and \( A \subset X \) satisfying (C1). Then \( (X, \tau) \) is semi-compact (resp s-closed) iff \( (X, \tau[A]) \) is semi-compact (resp s-closed).

PROOF. By Theorem 2.2, the proof is obvious.

THEOREM 2.6. If \( X \) is a space, and \( A \subset X \), and \( (X, \tau[A]) \) is S-closed (resp. s-closed), then \( (X, \tau) \) is S-closed (resp. s-closed).

PROOF. Since \( SO(X, \tau) \subset SO(X, \tau[A]) \), the proof is obvious.

3. \( L - T_i \) AND Q - L - T_i SPACES

Let \( R \) be a topological property which is preserved under expansions.

DEFINITION 3.1. A topological space \( (X, \tau) \) is called \( L - R \) if there exists a subset \( S \subset X \) and \( S \neq X \), such that \( (X, \tau[S]) \) has \( R \).

PROPOSITION 3.1. If \( \tau \subset \tau' \), then for any \( S \subset X, \tau[S] \subset \tau'[S] \).

REMARK 3.1. If \( \tau \subset \tau' \) and \( \tau \) is \( L \ R \), then \( \tau' \) is also \( L \ R \), i.e. any expansion of \( L \ R \) topology on \( X \) is also \( L \ R \).

DEFINITION 3.2. Let \( i = 1, 2, 2.5 \) and \( j = 0, 1, 2, 2.5 \). We say that \( (X, \tau) \) is \( Q - L \ T_i \), if it is \( L - T_i \) and \( T_j \), where \( j < i \).

Now we are going to show that some of the properties \( L - T_i \) and \( Q - L - T_i \) are satisfied for some spaces but not for some other spaces.

PROPOSITION 3.2. For a space \( X \), the following diagram is easily obtained.

\[ T_2 \Rightarrow Q - L - T_2 \Rightarrow T_2 \Rightarrow Q - L - T_2 \Rightarrow T_1 \Rightarrow Q - L - T_1 \Rightarrow T_0. \]

EXAMPLE 3.1. Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, X, \{a, b\}, \{c, d\}\} \) is not \( T_0 \) if \( A = \{a, c\} \), then \( \tau[A] = \{\phi, X, \{b\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, d\}\} \) is \( T_0 \). This example is \( Q - L - T_0 \).

The following is an example of a \( Q - L - T_{2.5} \) but not \( T_{2.5} \).

EXAMPLE 3.2. Let \( X = N \times Z \cup \{(0, 0), (0, 1)\} \) where \( N \) is the natural numbers and \( Z \) the integers. The topology has as its base sets the following forms:

\[ U_n((a, 0)) = \{(a, 0)\} \cup \{(a, m)\} \mid m \geq n, \ n \in N \]

\[ U_n((-1, 1)) = \{(-1, 1)\} \cup \{(a, m)\} \mid a \geq n, m > 0, \ n \in N \]

\[ U_n((-1, -1)) = \{(-1, -1)\} \cup \{(a, m)\} \mid a \geq n, m < 0, \ n \in N \].

This space is \( T_2 \) but not \( T_{2.5} \) as \((-1, 1)\) and \((-1, -1)\) do not have disjoint closed neighborhoods.

Choosing \( A = N \times (Z \setminus \{0\}) \), the discrete expansion is the discrete topology and thus \( T_2 \).

EXAMPLE 3.3. Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, X, \{b\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}\} \), then \( \tau[A] = \text{Discrete} \). This example is \( Q - L - T_1 \) but not \( T_1 \) and is an example of a space which is not \( Q - L - T_2 \).

EXAMPLE 3.4. Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, X, \{a, b\}\} \). If \( A = \{a, b\} \), then \( \tau[A] = \text{Discrete} \) This example is not \( Q - L - T_1 \).

The excluded point topology on an infinite set \( X \) is the family consisting of \( \phi \) and all subsets of \( X \) not containing a point \( p \) of \( X \).

EXAMPLE 3.5. The excluded point topology is \( L - T_1 \) and not \( L - T_2 \) (also is an example of \( Q - L - T_1 \) but not \( T_1 \)).

PROOF. If \( X \) is an infinite set and \( p \) is the excluded point and \( A \subset X \), then:

(i) If \( p \notin A \), we have \( \tau[A] = \tau \cup \{X - B : B \subset A\} \). Thus \( \tau[A] \) is \( T_1 \) but not \( T_2 \).
(ii) If \( p \in A \), then \( A \) is closed, and there are two cases

(a) If \( B \subset A \), \( p \in B \) in this case any open set in \( \tau[A] \) is open in \( \tau \), i.e \( \tau = \tau[A] \)

(b) If \( B \subset A \), \( p \notin B \) as (i) Thus \( \tau[A] = \tau \cup \{X - B : B \subset A\} \)

**EXAMPLE 3.6.** Let \( X = [0, 1] \) and \( \tau = \{\phi, X, A \subset X : X - A \text{ is finite}\} \). If we take \( S = (0, 1] \), then \( \tau[S] \) is the Discrete space. This example is \( Q - L - T_2 \) but not \( T_2 \)

**THEOREM 3.1.** \( (X, \tau) \) is cid space iff \( \tau = \tau[A] \) whenever \( A \) is a countable infinite subset of \( X \)

**PROOF.** We assume that \( (X, \tau) \) is cid, then \( A \) is closed and discrete subspace. By Theorem 19 we have that \( \tau = \tau[A] \) Conversely we assume that \( \tau = \tau[A] \) By Theorem 8, we have that \( (A, \tau \cap A) \) is a discrete subspace of \( X \) and \( (X, \tau) \) is cid space

**THEOREM 3.2.** Every space \( (X, \tau) \) is \( L - T_0 \)

**PROOF.** Assume that \( x_0 \in X \) We aim to prove that \( \tau[X - \{x_0\}] \) is \( T_0 \). For this purpose let \( x, y \in X, x \neq y \), if \( U \in \tau \) is an open set containing \( x \), then \( U - \{y\} \) is an open set in \( \tau[X - \{x_0\}] \) and not containing \( y \). If \( x_0 = x \), then \( X - \{y\} \) is an open in \( \tau[X - \{x_0\}] \) and not containing \( y \). This completes the proof

The following example illustrates a \( Q - L - T_2 \) space but not \( T_2 \)

**EXAMPLE 3.7.** (Countable complement topology [16]) If \( X \) is an uncountable set, we define the topology of countable complements on \( X \) by declaring open all sets whose complements are countable, together with \( \phi \) and \( X \). \((X, \tau) \) is \( T_1 \) but not \( T_2 \). Let \( A \subset X \) such that \( X - A \) is countable. For \( x_0 \in X - A \), \( A \cup \{x_0\} \) is \( \tau \)-open, and so \( (A \cup \{x_0\}) - A = \{x_0\} \in \tau[A] \). For \( x_0 \in A \), \( A \) is \( \tau \)-open, which means that \( A - (A - \{x_0\}) = \{x_0\} \) is \( \tau[A] \)-open. Thus \( \tau[A] \) is discrete and consequently \( T_2 \)

**UNSOLVED PROBLEM.** If \( (X, \tau) \) is a space which does not have a property \( P \), what are the properties of the subset \( A \) that make \( (X, \tau[A]) \) have \( P \) (for \( P \) fixed property)

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REFERENCES


