A NEW CRITERION FOR STARLIKE FUNCTIONS

LING YI
Department of Mathematics
Harbin Institute of Technology
Harbin, P R. China

SHUSEN DING
Department of Mathematics
The Florida State University
Tallahassee, Florida, 32306-3027

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ABSTRACT. In this paper we shall get a new criterion for starlikeness, and the hypothesis of this criterion is much weaker than those in [1] and [2].

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1. INTRODUCTION AND PRELIMINARIES.

Let $A$ be the class of functions $f(z)$, which are analytic in the unit disc $D = \{ z : |z| < 1 \}$, with $f(0) = f'(0) = 1 = 0$. Let $S$ be the set of starlike functions, $S = \{ f(z) \in A, \; Re(zf'(z)/f(z)) > 0, \; z \in D \}$.

R. Singh and S. Singh in [1] proved that if $f(z) \in A$ and $Re[f'(z) + zf''(z)] > 0$, $z \in D$, then $f(z) \in S$.

Recently, R. Singh and S. Singh in [2] proved that if $f(z) \in A$ and $Re[f'(z) + zf''(z)] > -\frac{1}{4}$, $z \in D$, then $f(z) \in S$.

In this paper we shall show that the assertion of R. Singh and S. Singh holds under a much weaker hypothesis.

Lemma 1. Suppose that the function $\psi: C \times C \to C$ satisfies the condition $Re(iz, y; z) < \delta$ for all real $z, y \leq \frac{1 + \epsilon z^2}{2}$ and all $z \in D$. If $p(z) = 1 + p_1 z + \cdots$ is analytic in $D$ and

$$Re(p(z), zp'(z); z) > \delta, \; \text{for} \; z \in D$$

then $Re(p(z)) > 0$ in $D$.

A general form of this lemma can be found in [3]. In [4] the authors got the following result.

Lemma 2. Let $\alpha > 0, \beta < 1$. If the function $p$ is analytic in $D$, with $p(0) = 1$ and

$$Re[p(z) + \alpha zp'(z)] > \beta, \; z \in D$$

then $Re(p(z)) > (2\beta - 1) + 2(1 - \beta)F(1, \frac{1}{\alpha}; \frac{1}{\alpha} + 1; -1)$, $z \in D$, where $F(a, b, c; z)$ is a hypergeometric function. This result is sharp.

By taking $\alpha = 1$ in lemma 2, we obtain

Lemma 3. Let $\beta < 1$. If the function $p$ is analytic in $D$, with $p(0) = 1$ and

$$Re[p(z) + zp'(z)] > \beta, \; z \in D$$

then $Re(p(z)) > (2\beta - 1) + 2(1 - \beta)ln 2, \; z \in D$, and the result is sharp.
2. MAIN RESULT

THEOREM. If \( f(z) \in \mathcal{A} \) and

\[
\text{Re} \left[ f'(z) + zf''(z) \right] > 1 - \frac{3}{4(1 - \ln 2)^2} + 2 \approx -0.263, \quad z \in D
\]

then \( f(z) \in S \).

PROOF. By using lemma 3, from (1) we have

\[
\text{Re} \left( f'(z) \right) > 1 - \frac{3(1 - \ln 2)}{2(1 - \ln 2)^2} + 1 > 0, \quad z \in D.
\]

From (2) and lemma 3, we have

\[
\text{Re} \left( \frac{f(z)}{z} \right) > -2 + \frac{3}{2(1 - \ln 2)^2} \approx 0.526, \quad z \in D.
\]

Now, we let \( p(z) = zf'(z)/f(z) \) and \( \lambda(z) = f(z)/z \), then \( p(z) \) is analytic in \( D \) and \( p(0) = 1, \text{Re} \{ \lambda(z) \} > -2 + \frac{3}{2(1 - \ln 2)^2} \). A simple computation shows that

\[
f'(z) + zf''(z) = \lambda(z)[p^2(z) + 2p'(z)] = \psi(p(z), zp'(z); z),
\]

where \( \psi(u, v; z) = \lambda(z)(u^2 + v) \). Using (1), we have \( \text{Re} \{ \psi(p(z), zp'(z); z) \} > 1 - \frac{3}{4(1 - \ln 2)^2 + 2} \) for each \( z \in D \).

Now for all real \( x, y < -1/2(1 + x^2) \), we have

\[
\text{Re} \left[ \psi(iz, y; z) \right] = (y - x^2) \text{Re} \{ \lambda(z) \} \leq -\frac{1}{2}(1 + 3x^2) \text{Re} \{ \lambda(z) \} \leq -\frac{1}{2} \text{Re} \{ \lambda(z) \}
\]

for each \( z \in D \). Note that \( \text{Re} \{ \lambda(z) \} > -2 + \frac{3}{2(1 - \ln 2)^2} \), from (4) we get

\[
\text{Re} \left[ \psi(iz, y; z) \right] \leq 1 - \frac{3}{4(1 - \ln 2)^2 + 2}
\]

for all \( z \in D \). Thus by lemma 1, \( \text{Re} \{ p(z) \} > 0 \) in \( D \), that is, \( f(z) \in S \).

REMARK. For \( \beta < 1 \), let \( R(\beta) = \{ f \in \mathcal{A} : \text{Re} \left[ f'(z) + zf''(z) \right] > \beta, \quad z \in D \} \). It was proved in [4] that if \( f(z) \in R(\alpha_0) \) ( \( \alpha_0 = \frac{3}{2(1 - \ln 2)} \approx -0.61 \)), then \( f(z) \) is univalent, and the constant \( \alpha_0 \) can not be replaced by any less one. Our present theorem yields \( R \left( 1 - \frac{3}{4(1 - \ln 2)^2} \right) \subset S \). Thus, a natural problem which arises is to find \( \inf \{ \beta : R(\beta) \subset S \} \).

REFERENCES


