ON THE SURJECTIVITY OF LINEAR TRANSFORMATIONS

M. DAMLAKHI and V. ANANDAM
Department of Mathematics
King Saud University
P O Box 2455, Riyadh 11451, SAUDI ARABIA

(Received March 2, 1993 and in revised form February 11, 1993)

ABSTRACT. Let $B$ be a reflexive Banach space, $X$ a locally convex space and $T : B \to X$ (not necessarily bounded) linear transformation. A necessary and sufficient condition is obtained so that for a given $v \in X$ there is a solution for the equation $Tu = v$. This result is used to discuss the existence of an $L^p$-weak solution of $Du = v$ where $D$ is a differential operator with smooth coefficients and $v \in L^p$.

KEY WORDS AND PHRASES: Admissible linear operators, $L^p$-functions, harmonic functions

1991 AMS SUBJECT CLASSIFICATION CODES: 47F05, 47B38

1. INTRODUCTION

Let $T$ be a (not necessarily bounded) linear operator from a reflexive Banach space $B$ into a locally convex space $X$. We obtain a necessary and sufficient condition for the existence of a solution $u \in B$ to the equation $Tu = v$, when $v \in X$ is known.

In this context the following question arises naturally. Let $\Omega$ be an open set in $\mathbb{R}^n$ and $D$ a differential operator of order $m$ with $c^m$-coefficients in $\Omega$. Given $v \in L^p(\Omega)$ does $Du = v$ have a weak solution $u \in L^p(\Omega)$?

When $p = 2$, $\Omega$ is a bounded domain and $D$ has constant coefficients, L. Hörmander (to see Corollary 14, M. Schechter [1]) has proved that $Du = v$ has always a weak solution. The proof depends heavily on Hilbert space techniques as applied to $L^2(\Omega)$. Our investigation here is around the form the above result of Hormander takes when only Banach space methods are available as in $L^p(\Omega)$.

2. ADMISSIBLE LINEAR OPERATORS

Let $B$ be a Banach space and $X$ be a locally convex space. Let $B'$ and $X'$ denote the algebraic duals of $B$ and $X$, $B^*$ and $X^*$ denote their topological duals.

Given $T : B \to X$, a linear operator not necessarily bounded, define the linear operator $T^* : X' \to B'$ as follows. For $f \in X'$ and $x \in B$, $T^* f(x) = \langle x, T^* f \rangle = \langle Tx, f \rangle$.

**Lemma 1.** Let $B$ be a reflexive Banach space and $X$ be a locally convex space. $T : B \to X$ is a linear operator, not necessarily bounded. Suppose that there exists a subspace $H \subset X'$ such that $T^* (H) \subset B^*$. Then given $v \in X$, there exists $u \in B$, $\|u\| \leq c$ such that $\langle Tu, f \rangle = \langle v, f \rangle$ for every $f \in H$ if and only if $\|\langle v, f \rangle\| \leq c\|T^* f\|$

**Proof.** Let $\langle Tu, f \rangle = \langle v, f \rangle$ with $\|u\| \leq c$ and $f \in H$. Then $|\langle v, f \rangle| = |\langle u, T^* f \rangle| \leq \|u\| \|T^* f\| \leq c\|T^* f\|$

Conversely, define the linear functional $S$ on the subspace $T^* (H)$ so that, for $g \in T^* (H)$, $Sg = \langle v, f \rangle$ where $g = T^* f$ for some $f \in H$.

$S$ is well-defined, for, if $g = T^* f_1$, for some other $f_1 \in H$, then $|\langle v, f \rangle - \langle v, f_1 \rangle| = |\langle v, f - f_1 \rangle| \leq c\|T^* (f - f_1)\| = 0$. 

\[ \text{\|u\|} \leq c\|T^* f\| \leq c\|T^* f\| \]
It is clear that $S$ is a bounded linear functional on the subspace $T^*(H) \subset B^*$ with $\|S\| \leq c$ and hence by Hahn-Banach theorem extends as a bounded linear functional on $B^*$, preserving the norm.

This implies, since $B$ is reflexive, that there exists $u \in B$ such that for every $h \in B^*$, $\langle u, h \rangle = Sh$ and $\|u\| = \|S\| = c$.

In particular, if $h = T^*f$, $f \in H$, we have $\langle u, T^*f \rangle = S(T^*f) = \langle v, f \rangle$

Thus, for any $f \in H$, $\langle v, f \rangle = \langle u, T^*f \rangle = \langle Tu, f \rangle$

This completes the proof of the lemma.

**REMARK 2.1.** The above lemma is inspired from section 16 of M Schechter [1] where the existence of a weak solution of a differential operator in the Hilbert space $L^2(\Omega)$ is investigated.

**DEFINITION 2.1.** Let $B$ be a Banach space and $X$ be a locally convex space. A linear operator $T : B \to X$, is said to be admissible if there exists a weak*-dense subspace $M \subset X^*$ such that $T^*(M) \subset B^*$.

**PROPOSITION 2.1.** Let $B$ be a Banach space and $X$ be a Fréchet space. Let $T : B \to X$ be a linear operator. Then $T$ is continuous if and only if $T$ is admissible.

**PROOF.** If $T$ is continuous, then for any $f \in X^*$ clearly $T^*f \in B^*$ and hence $T$ is admissible.

Conversely, let $T$ be admissible with $T^*(M) \subset B^*$ where $M$ is a weak*-dense subspace of $X^*$. We will prove that $T$ is continuous by showing that $T$ is closed (W Rudin [2], p 50).

Let $x_n \in B$ be a sequence such that $\lim_n x_n = x$ and $\lim_n T x_n = y$. Then for any $f \in M$, $\langle x_n, T^*f \rangle = \langle Tx_n, f \rangle$.

Taking limits $\langle x, T^*f \rangle = \langle y, f \rangle$ which implies that $\langle Tx, f \rangle = \langle y, f \rangle$ for every $f \in M$ and consequently $\langle Tx, h \rangle = \langle y, h \rangle$ for every $h \in X^*$, since $M$ is $W^*$-dense in $X^*$.

This implies that $Tx = y$ since $X^*$ separates $X$, that is, $T$ is closed.

**THEOREM 2.1.** Let $B$ be a reflexive Banach space and $X$ be a locally convex space. Let $T : B \to X$ be an admissible linear operator with $T^*(M) \subset B^*$. Then, for any given $v \in X$, there exists $u \in B$ such that $\|u\| \leq c$ and $Tu = v$ if and only if $\|\langle v, f \rangle\| < c\|T^*f\|$ for every $f \in M$.

**PROOF.** In view of Lemma 1 (where we take $H = M$), it is enough to prove that the condition $\langle Tu, f \rangle = \langle v, f \rangle$ for every $f \in M$ is equivalent to the fact that $Tu = v$.

Now, the condition above is equivalent to the fact $\langle Tu, h \rangle = \langle v, h \rangle$ for every $h \in X^*$, since $M$ is dense in $X^*$ with its $W^*$-topology.

Since $X^*$ separates points on the locally convex space $X$, the latter condition is equivalent to the fact $Tu = v$.

**3. WEAK SOLUTIONS IN $L^p(\Omega)$**

Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 1$. Let $A = \sum_{|k| \leq m} a_k(x)D^k$ be a differential operator of order $m$, with $a_k(x) \in C^m(\Omega)$. Let $A^*$ denote the adjoint operator. Let $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

**THEOREM 3.1.** With the above assumptions on $A$ and $p$, let $f \in L^p(\Omega)$ be given. Then there exists a weak solution of $Au = f$, $u \in L^p(\Omega)$ and $\|u\|_p \leq c$ if and only if $\|\langle \phi, f \rangle\| \leq c\|A^*\phi\|_q$ for all $\phi \in C_0^\infty(\Omega)$.

**PROOF.** Suppose $f \in L^p$ and $Au = f$ has a weak solution $u \in L^p$, $\|u\| \leq c$.

Define, for $\phi \in C_0^\infty(\Omega)$ and $g \in L^p(\Omega)$, $\langle \phi, g \rangle = \int_\Omega \phi(x)g(x)dx$

Then, $\|\langle \phi, f \rangle\| = \|\langle \phi, Au \rangle\| = \|\langle A^*\phi, u \rangle\| \leq \|u\|_p\|A^*\phi\|_q \leq c\|A^*\phi\|_q$.

Conversely, define the linear functional $S$ on the subspace $A^*(C_0^\infty(\Omega))$ such that $S(A^*\phi) = \langle \phi, f \rangle = \int_\Omega \bar{f}\phi dx$.

Then, as in Lemma 1, $S$ is a well-defined linear functional on $A^*(C_0^\infty(\Omega)) \subset L^q(\Omega)$ with $\|S\| \leq c$ and hence extends as a continuous linear functional on $L^q(\Omega)$, so that there exists $u \in L^p(\Omega)$ satisfying the condition $S(v) = \langle v, u \rangle$ for all $v \in A^*(C_0^\infty(\Omega))$ and $\|u\|_p = \|S\| \leq c$. 

In particular, for any \( \phi \in C_0^\infty(\Omega) \), \( \langle \phi, f \rangle = S(A^*\phi) = \langle A^*\phi, u \rangle = \langle \phi, Au \rangle \) Hence \( u \) is a weak solution of \( Au = f \)

**THEOREM 3.2.** Let \( f \in L^1_{loc}(\Omega) \) Then there exists a bounded weak solution \( u \) of the equation \( Au = f \) if and only if \( \int_{\Omega} f(x) \phi(x) dx \leq C\|A^*\phi\|_1 \) for every \( \phi \in C_0^\infty(\Omega) \)

**PROOF.** In view of the above theorem, we will give here only a few details of the proof

On \( A^*(c_0^\infty(\Omega)) \), considered as a subspace of \( L^1(\Omega) \), define the linear functional \( S \) such that \( S(A^*\phi) = \langle \phi, f \rangle = \int f(x) dx \). Then \( S \) extends as a bounded linear functional on \( L^1(\Omega) \) so that there exists \( u \in L^1(\Omega) \) such that \( Sg = \langle g, u \rangle \) for every \( g \in L^1(\Omega) \)

This leads to the fact that \( u \) is a weak solution of \( Au = f \)

In the context of the above theorem where we were looking for a bounded weak solution of a differential equation, the following proposition concerning the bounded solutions of the Laplacian in \( \mathbb{R}^n \) is of interest

**PROPOSITION 3.1.** Let \( f \in C^6_{c}(\mathbb{R}^n) \), having compact support \( K \), be given in \( \mathbb{R}^n \) Then, if \( n \geq 3 \), there always exists a bounded \( u \in C^\infty(\mathbb{R}^n) \) such that \( \Delta u = f \), if \( n = 1 \) or \( 2 \), such a bounded \( C^\infty \) solution exists if and only if \( \int_K f(x) dx = 0 \)

**PROOF.** Since \( \Delta \) is an elliptic differential operator with constant coefficients, there always exists some \( u \in C^\infty(\mathbb{R}^n) \) such that \( \Delta u = f \). Here we are looking for a bounded function \( u \) in \( C^\infty(\mathbb{R}^n) \)

Let

\[
E_n(x) = \begin{cases} 
|\frac{x}{n}| & \text{if } n = 1 \\
\log|\frac{x}{n}| & \text{if } n = 2 \\
\frac{1}{|\frac{x}{n}|^{n-2}} & \text{if } n > 3
\end{cases}
\]

Now, using the results in [2], we can show that for a fixed \( y \in K \) and any \( x \in K^c \),

\[
u(x) = \left( \int f(x) dx \right) \beta_1 E_n(x - y) + l(x)
\]

where \( l(x) \) is a bounded harmonic function in \( K^c \) if \( n \geq 2 \) (and affine bounded if \( n = 1 \))

Here \( \beta_1 = \frac{1}{2}, \beta_2 = \frac{1}{2n} \) and \( \beta_n = \frac{1}{(n-2)\alpha_0} \), \( \alpha_n \) being the measure of the unit sphere in \( \mathbb{R}^n \)

Consequently, using the fact that \( E_n(x) \) is bounded in a neighborhood of the point at infinity if and only if \( n \geq 3 \), we arrive at the conclusion of the proposition

**NOTE.** Since a bounded harmonic function outside a compact set in \( \mathbb{R}^n \), \( n \geq 2 \), tends to a limit at infinity, if \( u \) is a bounded solution of \( \Delta u = f \in c_0^\infty \), we can choose \( u_0 \in C^\infty(\mathbb{R}^n) \) so that \( u_0 \) tends to 0 at infinity and satisfies the condition \( \Delta u_0 = f \). In this case, \( u_0 \) is unique

4. **SURJECTIVITY ON THE SOBOLEV SPACES**

We conclude this article with a remark on the solutions of a differential operator on the Sobolev spaces \( H^s(\mathbb{R}^n) \)

We make use of the following properties.

i) For each real \( s \), \( H^s(\mathbb{R}^n) \) is a Hilbert space such that \( H^s \subset H^t \) if \( t \leq s \)

ii) \( H^s \) is the completion of \( c_0^\infty \) in the norm \( \| \cdot \|_{H^s} \)

iii) For any \( s \), \( H^{-s} \) represents the topological dual of \( H^s \)

iv) If \( s > \frac{n}{2} + k \) where \( k \) is a nonnegative integer, then \( H^s \subset C^k \)

v) If \( A \) is a differential operator of order \( m \) with \( c^m \)-coefficients, \( A^*(c_0^\infty) \subset L^2 \subset H^s \) for any \( s \geq 0 \)

vi) If \( A \) is a differential operator of order \( m \) with \( c^m \)-coefficients, \( A^*(c_0^\infty) \subset H^s \) for every real \( s \)

Then, with arguments similar to those utilized to prove some of the earlier results, we obtain
**THEOREM 4.1.** Let $T$ be a distribution in $\mathbb{R}^n$, $n \geq 1$ Suppose that $A$ is a differential operator of order $m$ satisfying one of the following two sets of assumptions

a) $A$ has $c^m$-coefficients and $s \geq 0$

b) $A$ has $c^s$-coefficients and $s$ is any real number

Then $T = Au$ in the sense of distribution, for some $u \in H^s$, if and only if $|T(\phi)| \leq c\|A^*\phi\|_H$ for all $\phi \in c_0^s(\mathbb{R}^n)$

**REMARK 4.1.** Let $A$ be a differential operator of order $m$ with coefficients either constants or from the Schwartz's space (i.e., rapidly decreasing $c^\infty$-functions). Then if $|T(\phi)| \leq c\|A^*\phi\|_H$ for all $\phi \in c_0^s$, we have as in the above theorem, $T = Au$, $u \in H^s$

But, in this special case, $T \in H^s$ $m$ and consequently, if $s > \frac{3}{2} + m$, then $T = Au$ in the classical sense i.e., $u$ is a strong solution of the differential equation.

$H^k(\Omega)$-spaces Let now $\Omega$ be an open set in $\mathbb{R}^n$, $n \geq 1$. Recall that for any positive integer $k$, $H^k_0(\Omega)$ is defined as the closure of $c_0^\infty(\Omega)$ in $H^k(\Omega)$. For any $v \in L^2(\Omega)$, define $\|v\|_{-k} = \sup_{u \in H^k_0(\Omega)} \frac{|(v,u)|}{\|u\|_{H^k(\Omega)}}$.

Then, if $H^{-k}(\Omega)$ denotes the completion of $L^2(\Omega)$ in the norm $\|\cdot\|_{-k}$, $H^{-k}(\Omega)$ is the topological dual of $H^k_0(\Omega)$ for any integer $k \geq 0$ (see Al-Gwaiz [4], p 191).

With this background, we can state an analogue of Theorem 4.1 as follows.

**THEOREM 4.2.** Let $T$ be a distribution in an open set $\Omega$ in $\mathbb{R}^n$, $n \geq 1$. Suppose $A$ is a linear differential operator with $c^\infty(\Omega)$-coefficients. Then, for any integer $k \geq 0$, there exists $u \in H^{-k}(\Omega)$ such that $Au = T$ if and only if $|T\phi| \leq c\|A^*\phi\|_{H^k(\Omega)}$, for all $\phi \in c_0^\infty(\Omega)$.

**REFERENCES**


