INTEGRATED COSINE FUNCTIONS

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ABSTRACT. In order to the second order Cauchy problem (CP2): \( x''(t) = Ax(t), x(0) = x \in D(A^n), x'(0) = y \in D(A^m) \) on a Banach space, Arendt and Kellermann recently introduced the integrated cosine function. This paper is concerned with its basic theory, which contain some properties, perturbation and approximation theorems, the relationship to analytic integrated semigroups, interpolation and extrapolation theorems.

KEY WORDS AND PHRASES. Integrated cosine function, analytic integrated semigroup, perturbation, approximation, interpolation, extrapolation.

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1. BASIC PROPERTIES

Let \( A \) be a linear operator on a Banach space \( X \). If there exist \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( M, \omega > 0 \) and a strongly continuous family \( C(t) \) in \( L(X) \) with \( \|C(t)\| \leq Me^{\omega t} \) for \( t \geq 0 \) such that \( \omega^2, \omega \in \rho(A) \) and \( R(\lambda^2, A) = \lambda^{n-1} \int_0^\infty e^{-\lambda t} C(t)dt \) for \( \lambda > \omega \), then we say that \( A \) generates the (exponentially bounded) \( n \)-times integrated cosine function \( C(t) \) (see [1]), and write \( (A, C(t)) \in G_n(M, \omega, X) \). Set \( G_n(M, \omega, X) = \cup_{t \geq 0} G_n(M, \omega, X) \). The definition of integrated semigroups see e.g. [6], and the corresponding notations \( G_n(M, \omega, X) \) and \( \mathcal{G}_n(X) \) also can be introduced.

It is known that a 0-times integrated cosine function consists with a cosine function, while the following relationship can be shown by the same method in [4, Th.3.5].

**Proposition 1.1.** \( (A, C(t)) \in G_n(M, \omega, X) \) iff \( (A, C(t)) \in \mathcal{G}_{n+1}(X^2) \), where \( A = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \)

\[ C(t) = \left( \begin{array}{c} \int_0^t C(s)ds \\ C(t) - t^n I/n! \int_0^t C(s)ds \end{array} \right) \text{ for } t \geq 0. \]

The basic properties of integrated cosine functions now can be deduced from Prop.1.1, the properties of Laplace transforms and integrated semigroups (cf. [6]). We omit the details.

**Proposition 2.** Let \( (A, C(t)) \in G_n(M, \omega, X), \langle k \rangle = \{k+1\}, \text{ and } t, s \in \mathbb{R}^+ \equiv [0, \infty) \). Then

(a) \( C(t)C(s) = C(s)C(t), C(t)x = 0 \) \( (r \geq 0) \) implies \( x = 0 \), and \( C(t) \) is uniquely determined by \( A \).

(b) \( (A, \int_0^t (t-s)^k C(s)ds/k!) \in G_{n+k+1}(M\omega^{-k-1}, \omega, X) \) for \( k \in \mathbb{N}_0 \).

(c) \( R(\lambda^2, A) = \lambda^{n-1} \int_0^\infty e^{-\lambda t} C(t)dt \) and \( \|R(\lambda^2, A)\| \leq M|\lambda|^{n-1}(\text{Re}\lambda - \omega)^{-1} \) for \( \text{Re}\lambda > \omega \).

(d) For \( x \in D(A^{(k)}), C(\cdot)x \in C^k(\mathbb{R}^+, X) \) \( (k \in \mathbb{N}_0) \) and \( \lambda^{n+k} R(\lambda^2, A)x \rightarrow C^{(k)}(0) = \delta_k x \) as \( \lambda \rightarrow \infty \) \( (0 \leq k \leq n) \), where \( \delta \) denotes the Kronecker delta.

(e) \( C(t)x \in D(A), \int_0^t (t-s)C(s)ds \in D(A) \) and \( A \int_0^t (t-s)C(s)ds = C(t)x - t^n x/n! \) for \( x \in X \); \( AC(t)x = C(t)Ax \) and \( C(t)x = \int_0^t (t-s)C(s)Axds + t^n x/n! \) for \( x \in D(A) \).

(f) \( \int_0^t C(s)ds \in D(A) \) and \( A \int_0^t C(s)ds = C(t)x - t^{n-1}x/(n-1)! \) for \( x \in X_1 \); \( C(t)x \in D(A) \) and \( AC(t)x = C^n(t)x - t^{n-2}x/(n-2)! \) for \( x \in X_2 \), where \( X_1 = \{x \in X; C(\cdot)x \in C^0(\mathbb{R}^+, X)\} \).
(g) $C(t)S(s) + S(t)C(s) = \{f_{t+r}^{s+t} - f_0^r\}(s + t - r)^{n-1}S(r)dr/(n-1)!$, where $S(t) = f_0^t C(s)ds$.

We conclude this section with some remarks on CP2: $x(t)$ is called a solution of CP2 if $x(\cdot) \in C^2(R^+, X) \cap C(R^+, D(A))$ satisfying CP2. CP2 is called $n$-wellposed if CP2 has a unique solution $x(t)$ for every $(x,y) \in D(A^{(n+2)}) \times D(A^{(n+1)})$ satisfying $\|x(t)\| \leq p(t)(\|x\|_{(n)} + \|y\|_{(n-1)})$ for some locally bounded function $p(t)$. CP2 is called exponentially $n$-wellposed if, in addition, $p(t) = M e^{\omega t}$.

The following result follows from Prop.1.1, [3, Th.2.5] and [6, 7]. The details are omitted.

THEOREM 1.3. Let $\omega > 0$, $n, m \in N_0$ and $\rho(A) \neq \emptyset$. Then the following statements hold.

(a) If CP2 has a unique solution for every $(x,y) \in D(A^{(n+2)}) \times D(A^{(n+1)})$ then CP2 is $k$-wellposed where $k = \max(m, n)$. In particular, CP2 has a unique solution for every $(x,y) \in D(A^{(n+2)}) \times D(A^{(n+1)})$ iff CP2 is $n$-wellposed.

(b) $A \in G_n(\omega, X)$ iff CP2 has a unique solution $x(t)$ with $\|x^n(t)\| = O(e^{\omega t})$ for every $(x,y) \in D(A^{(n+2)}) \times D(A^{(n+1)})$. If, in addition, $\overline{D(A)} = X$ then $A \in G_n(\omega, X)$ iff CP2 is exponentially $n$-wellposed.

2. PERTURBATIONS

We first consider the perturbation problem of $2m$-times integrated cosine functions. The following is a generalization of the Takenaka-Okazawa theorem (cf. [10, II]).

THEOREM 2.1. Let $(A, C(t)) \in G_{2m}(X)$ and $B$ be a linear operator on $X$ satisfying

\((b_1)\) $D(A) \subset D(B)$, $R(B) \subset D(A_{12m}^{+})$ and $BR(\lambda_0, A) \in L(X)$ for some $\lambda_0 \in \rho(A)$, where $R(B)$ denotes the range of $B$, and $A_{1} = A|_{D(A)}$ (the part of $A$ in $D(A)$).

\((b_2)\) $A_{12m} \equiv \lim_{\lambda \to \infty} \lambda \leq \omega$, where $\lambda_1 \equiv \sup\{\int_0^{\infty} e^{-\lambda t}R((2m-1)(t))x_{m}dt; x \in D(A^{(m+1)}), \|x\|_{m} \leq 1\}$ and $\|x\|_{m} = \|x\|_{A_{12m}^{+}} = \sum_{j=0}^{m} \|A^jx\|_{m}$ for $x \in D(A^{(m+1)})$.

Then $A + qB \in G_{2m}(D(A))$ for every $q$ with $|q| < 1/l_{\infty}$.

PROOF. Let $A = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$ and $C(t)$ be given by (1) with $n = 2m$. Then

\[ R(\lambda, A) = \begin{pmatrix} \lambda & I \\ A & \lambda \end{pmatrix} \] for $\lambda \in D(A)$, (2)

and therefore (b1) implies (b1'): $D(A) \subset D(B)$, $R(B) \subset D(A_{12m+1}^{+})$ and $BR(\lambda_0^{1/2}, A) \in L(X)$. Since $\|x,y\|_{m} \equiv \|x\|_{A_{12m+1}^{+}} + \|y\|_{A_{12m+1}^{+}}$ is equivalent to $\|(x,y)\|^A_{2m+1}$ on $D(A_{2m+1}^{+})$ we have that

\[ l_{12m} \equiv \sup\{\int_0^{\infty} e^{-\lambda t}R((2m+1)(t))x_{m}dt; (x,y) \in D(A_{2m+2}^{+}), \|x,y\|_{m} \leq 1\} \leq \max\{l_{12m}, l_{1}^*\},\]

where $l_{12m} \equiv \sup\{\int_0^{\infty} e^{-\lambda t}R((2m)(t))x_{m}dt; x \in D(A_{m}^{+}), \|x\|_{m} \leq 1\}$. Noting that $l_{1} \leq l_{12m}^*$ and, by (b1), $l_{12m} \to 0$ as $\lambda \to \infty$, one has that $\lim_{\lambda \to \infty} l_{12m}^* = l_{\infty}$. Combining (b1'), Prop.1.1, [10, II, Th.4.2] with this yields that $A_{1} + qB \in G_{2m+1}(D(A))$ for $|q| < 1/l_{\infty}$. Let $C(t)$ generated by $A_{1} + qB$ have the form $C(t) \equiv S_{n}(t)/t^{(2m+1)!}$.

\[ C_{1}(t) = t^{(2m+1)!}/(2m+1)! \] Then we can deduce from (1) and (2) that

\[ \lambda^{2m-1} \int_0^{\infty} e^{-\lambda t}C_{1}(t)x_{m}dt = R(\lambda^{2m}, A_{1} + qB)x = \lambda^{2m} \int_0^{\infty} e^{-\lambda t}S_{n}(t)x_{m}dt \quad \text{for} \quad x \in D(A).\]

Consequently $C_{1}(t) \in L(D(A))$, and so the claim follows.

The subsequent theorem can be shown by [10, II, Th.4.3] and the same method as above.

THEOREM 2.2. Let $(A, C(t)) \in G_{2m}(X)$ and $B$ be a closed linear operator on $X$ satisfying

\((b_3)\) $D(A) \cup \int_0^{t} C(s)ds \subset D(D(A))$ for $t \geq 0$ and $R(D(A)) \subset D(A_{m}^{+})$.

\((b_4)\) $B \int_0^{t} C(s)ds \subset C(R^{+}, D(A_{m}^{+}))$ for $x \in X$ where $[D(A_{m}^{+})] = (D(A^{m}), \| \cdot \|_{m})$, and $l_{\infty} \equiv \lim_{\lambda \to \infty} \lambda \leq \omega$ where $\lambda_{1} \equiv \sup\{\int_0^{\infty} e^{-\lambda t}R((2m-1)(t))x_{m}dt; x \in D(A^{(m+1)}), \|x\|_{m} \leq 1\}$. Then $A + qB \in G_{2m}(X)$ for every $q$ with $|q| < 1/l_{\infty}$.

In the case $m = 0$, since $l_{\infty} = 0$ (see [9, Lemma]) and (b4) can be replaced by $\int_0^{t} C(s)ds \in D(D(B))$ for $t \geq 0$, Theorem 3.2 (with $m = 0$) is consistent with [9, Prop.].

We now turn to the perturbation problem of $(2m + 1)$-times integrated cosine functions.
THEOREM 2.3. Let \((A, C(t)) \in G_{2m+1}(X)\) and \(B\) be a linear operator on \(X\) satisfying (b1) and 
\[\|BC(t)x\| \leq Me^{\omega t}\|x\| \quad \text{for} \quad x \in D(A) \text{ and } t \geq 0.\] Then \(A + B \in G_{2m+1}(D(A)).\)

PROOF. We first note that \(\rho(A) \subset \rho(A_1)\) and \(R(\lambda, A_1) = R(\lambda, A)\) on \(D(A)\) for \(\lambda \in \rho(A)\).

Next, set \(C_1(t) = (\lambda_0 - A)^mBC(t)x\) and \(C_2(t) = (\lambda_0 - A_1)^mA(t)\) on \(D(A)\) for some \(\lambda_0 \in \rho(A)\). Then, by our assumptions and Prop. 1.2(d), \(C_1(t)\) can be extended to strongly continuous families in \(L(D(A))\), and satisfy \(BR(\lambda, A_1) = \lambda^m \int_0^\infty e^{-\lambda t}C(t)\) and \(B_2 = (\lambda_0 - A_1)^mB\). It follows thus that

\[C_3(t) \equiv C(t) + \{C'(t)x\} + \{I + C_1(t) + \sum_{k=2}^{\infty} C_1(t)^k\} \cdot C_2(t) \quad \text{for} \quad t \geq 0\]

is also a strongly continuous family in \(L(D(A))\), where \(\ast\) denotes the convolution. Now, one can directly check that \(R(\lambda^2, A + B) = \lambda^m \int_0^\infty e^{-\lambda t}C(t)\) for large \(\lambda\).

By a modification of the proof of Theorem 3.3 we can show the following

THEOREM 2.4. Let \((A, C(t)) \in G_{2m+1}(X)\) and \(B\) be a closed linear operator on \(X\) satisfying

\((b1) D(A) \cup C(t)(X) \subset D(B)\) for \(t \geq 0\) and \(R(B) \subset D(A^m)\).

\((b2) BC(t)x \in (C(R^+, [D(A^m)])\) and 
\[\|BC(t)x\| \leq Me^{\omega t}\|x\| \quad \text{for} \quad x \in X \text{ and } t \geq 0.\]

Then \(A + B \in G_{2m+1}(X)\).

As a generalization of [1, Th. 5.3] we have

COROLLARY 2.5. Let \(A \in G_n(X)\), and let \(B \in L(X)\) and \(B : D(A_0) \to D(A_0+1)\) for some \(\lambda_0 \in \rho(A)\). Then \(A + B \in G_n(X)\).

3. APPROXIMATIONS

The first approximation theorem is a direct consequence of [10, I, Th. 3].

THEOREM 3.1. Let \((A_k, C_k(t)) \in G_n(M, \omega, X)\) for some \(n \in N_0\). If \(\lim_{k \to \infty} R(\lambda^2, A_k)x = R(\lambda^2, A_0)x\) for \(x \in X\) and some \(\lambda_0 \in \rho(A)\), then \(\lim_{k \to \infty} R(\lambda^2, A_0)x = R(\lambda^2, A_0)x\) for \(x \in X\), uniformly for \(A\) in compact subsets of \(\rho(A)\).

THEOREM 3.2. Let \((A_k, C_k(t)) \in G_n(M, \omega, X)\) such that \(\lim_{k \to \infty} R(\lambda^2, A_k)x = R(\lambda^2, A_0)x\) for \(x \in X\) and some \(\lambda_0 \in \rho(A)\) and some injective operator \(R_0\). Then \((A, C(t)) \in G_{n+1}(M, \omega, X)\) such that \(\lim_{k \to \infty} R(\lambda^2, A_k)x = R(\lambda^2, A_0)x\) and \(R_0 = R(\lambda^2, A)\) for \(x \in X\) and some \(\lambda_0 \in \rho(A)\).

PROOF. For every \((A_k, C_k(t)) \in G_n(M, \omega, X)\) we define \((A_k, C_k(t)) \in G_{n+1}(\omega, X^2)\) as in Prop. 1.1. Thus (2) and our assumption implies that

\[\lim_{k \to \infty} \sup_{0 \leq t \leq T} \int_0^T C_k(s,x,y)ds = 0\]

Conversely, if \(D(A_0) = X\) and \(\lim_{k \to \infty} C_k(t)x = C(t)x\) for \(x \in X\) and \(t \geq 0\) then \(\lim_{k \to \infty} R(\lambda^2, A_k)x = R(\lambda^2, A_0)x\) for \(x \in X\), uniformly for \(\lambda\) in compact subsets of \(\rho(A)\).

THEOREM 3.3. Let \((A_k, C_k(t)) \in G_n(M, \omega, X)\) for some \(n \in N_0\), and \(\lim_{k \to \infty} R(\lambda^2, A_k)x = R(\lambda^2, A_0)x\) for \(x \in X\) and some \(\lambda_0 \in \rho(A)\) and some injective operator \(R_0\). Then \((A, C(t)) \in G_n(M, \omega, X)\) such that \(\lim_{k \to \infty} R(\lambda^2, A_k)x = R(\lambda^2, A_0)x\) and \(R_0 = R(\lambda^2, A)\) for \(x \in X\) and some \(\lambda_0 \in \rho(A)\).

PROOF. For every \((A_k, C_k(t)) \in G_n(M, \omega, X)\) we define \((A_k, C_k(t)) \in G_{n+1}(\omega, X^2)\) as in Prop. 1.1. Thus (2) and our assumption implies that

\[\lim_{k \to \infty} R(\lambda_0, A_k)x = R(\lambda_0, A_0)x\]

It follows from [10, I, Th. 4] that there exists \((A, C(t)) \in G_{n+1}(\omega, X^2)\) such that \(R_0 = R(\lambda_0, A)\) and

\[\lim_{k \to \infty} \sup_{0 \leq t \leq T} \|\int_0^T C_k(s,x,y)ds - C(t,x,y)\| = 0\]

It is easy to show, by \(R_0 = R(\lambda_0, A)\), that \(A = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\) and \(R_0 = R(\lambda_0, A)\), while the remainder conclusions now can be deduced from Prop. 1.1 and (4).

4. RELATIONSHIP TO INTEGRATED SEMIGROUPS

Let \(A\) be a linear operator on \(X\). If there exist \(n \in N_0, \omega > 0\) and a strongly continuous family \(T(t) \in L(X)\) for \(t > 0\) such that \(\omega \in \rho(A), \int_0^\infty e^{-\omega t}\|T(t)\|dt < \infty\) and \(R(\lambda, A) = \lambda^n \int_0^\infty e^{-\lambda t}T(t)dt\) for \(\lambda > \omega\), then we say that \(A\) generates the \(n\)-times semigroup \(T(t)\), and write
(A, T(t)) ∈ G_n(X). Obviously, (A, S(t)) ∈ G_n(X) implies (A, S(t)) ∈ G_n(X), and (A, T(t)) ∈ G_n(X) implies (A, ∫_0^t T(s)ds) ∈ G_{n+1}(X) (see [2, Prop.1.1]).

**Lemma 4.1.** Let (A, C(t)) ∈ G_n(X) and Σ_θ = \{t; |arg(t| < θ \} \{0\} (0 < θ < π/2). Define

\[ T_k(t) = \begin{cases} (-1)^k(πt)^{-1/2} \int_0^∞ [D^k_s \exp(-s^2/4t)] C(s)ds & \text{for } k ∈ \mathbb{N}_0 \\ (πt)^{-1/2} \int_0^∞ \exp(-s^2/4t) [∫_0^t C(r)dr] ds & \text{for } k = -1, \end{cases} \]  

(5)

where \( D_s = ∂/∂s \). Then \( T_k(t) \) is analytic and \( T_k(t) = T_{k+2}(t) \) for \( Re > 0 \) and \( k ∈ \mathbb{N}_0 \). If in addition \( ||C(t)|| ≤ M t^d e^{ωt} (t ≥ 0) \) for some \( d ≥ 0 \), then for \( k ∈ \mathbb{N}_0 \) and \( 0 < θ < π/2 \)

(a) \( ||T_k(t)|| ≤ M ||t||^{k+1/2} ||Re||^{-k-1} (1 + ||t||^2/Re)^{k+1/2} ||C(t)||^k ||t||^{1/2} \exp(ω ||t||/Re) \) for \( Re > 0 \).

(b) \( ||T_k(t)|| ≤ M ||t||^{k+1/2} ||Re||^{-k-1} (1 + ||t||^2/Re)^{k+1/2} ||C(t)||^k ||t||^{1/2} \exp(ω ||t||/Re) \) for \( Re > 0 \).

(c) \( ||T_{-1}(t)|| ≤ M ||t||^{k+3/2} ||Re||^{-k-1} (1 + ||t||^2/Re)^{k+1/2} ||C(t)||^k ||t||^{1/2} \exp(ω ||t||/Re) \) for \( Re > 0 \).

(d) \( ||T_{-1}(t)|| ≤ M ||t||^{k+3/2} ||Re||^{-k-1} (1 + ||t||^2/Re)^{k+1/2} ||C(t)||^k ||t||^{1/2} \exp(ω ||t||/Re) \) for \( Re > 0 \).

(e) In the case \(-1 ≤ k ≤ d \) we have that \( \int_0^∞ e^{-λt} ||T_k(t)|| dt < ∞ \) and

\[ R(λ, A) = λ^{(n-k)/2} \int_0^∞ e^{-λt} ||T_k(t)|| dt \]  

for \( λ > ω^2 \).

**Proof.** The analyticity of \( T_k(t) \) and \( T_k(t) = T_{k+2}(t) \) are easily shown. By (5) we have

\[ ||T_k(t)|| ≤ M ||t||^{-1/2} \int_0^∞ \sum_{i=0}^{k-1} s^{k-2i} ||t||^{-k} \exp(-s^2 Re/4||t||^2) s^{d+ω} ds \]

(6)

for \( Re > 0 \) and \( k ∈ \mathbb{N}_0 \), where \( M \) denotes a generic constant. Thus (a) is proved. (b) follows from (a) and \( Re > ||t|| \cos θ \) (cf. the proof of Lemma 4.1(a)). Since \( ||C(t)|| ≤ M t^d e^{ωt} \) implies \( ||∫_0^∞ C(s)ds|| ≤ M t^{d+1} e^{ωt} \), (c) and (d) follow from (a) and (b) (replacing \( k, d \) with \( 0, d+1 \)), respectively. Finally, \( \int_0^∞ e^{-λt} ||T_k(t)|| dt < ∞ \) follows from (b) and (d), while by \( T_k(t) = T_{k+2}(t) \) and Fubini’s theorem we can check (6).

**Corollary 4.2.** Let (A, C(t)) ∈ G_n(X) and \( ||C(t)|| ≤ M t^d e^{ωt} (t ≥ 0) \) for some \( 0 ≤ d ≤ n \).

If \( k = n - 2m \) and \( m \) is the least integer > \( (n - d - 2)/2 \), then \( (A, T(t)) ∈ G_n(X) \).

In the sequel, we infer to, e.g., [10, III] for analytic integrated semigroups, and write \( (A, S(t)) ∈ H_n(θ, ω, X) \) for \( A \) generating the analytic \( n \)-times integrated semigroup \( S(t) \) of type \( (θ, ω) \).

**Theorem 4.3.** Let the conditions of Corollary 5.3 be satisfied. Then for every \( θ ∈ (0, π/2) \)

(a) If \( k < d \) then \( (A, S_k(t)) ∈ H_n(θ, ω^2/\cos^2 θ, X) \), where \( S_k(t) = T_k(t) \) for \( t ∈ Σ_θ \), and \( S_k(0) = 0 \).

(b) In the case \( k = d \), if in addition (i) \( D(A) = X \), or (ii) \( \lim_{t→0} C(t)x/t^d (x ∈ D) \) exist where \( D = X \), then \( (A, S_k(t)) ∈ H_n(θ, ω^2/\cos^2 θ, X) \), where \( S_k(t) = T_k(t) \) for \( t ∈ Σ_θ \), and \( S_k(0) = 0 \).

**Proof.** (a) follows from Lemma 4.1. To prove (b), we note, by Prop.1.2, that (i) implies (ii) with \( D = D(A^{(k)}) \), and that (ii) and \( ||C(t)|| ≤ M t^d e^{ωt} \) imply \( \lim_{t→0} C(t)x/t^k = δ_{θn}x/n! \) for \( x ∈ X \). The following proof is divided into cases.

Case \( k < n \). Fix \( x ∈ X \). Then for arbitrary \( ε > 0 \), there exists \( δ > 0 \) such that \( ||C(s)x/s^k|| < ε \) \((0 < s < δ)\). Thus we have (cf. the proof of Lemma 4.1(a))

| ||T_k(t)|| | ≤ M ||t||^{-1/2} \sum_{i=0}^{k-1} s^{k-2j} ||t||^{-k} \exp(-s^2 Re/4||t||^2) s^{d+ω} ds

+ M ||t||^{-1/2} \sum_{i=0}^{k-1} \int_0^∞ s^{k-2j} ||t||^{-k} \exp(-s^2 Re/4||t||^2) s^{d+ω} ds

\[ ≤ M δ ε + M θ \exp(ω/|t|/|cos θ|) \sum_{i=0}^{k-1} \int_0^∞ (u^{2k-2i} + ||t||^{-k}) \exp(-u^2) du, \]

(7)

where \( γ(t) = δ√Re/|t| - |ωt|/√Re → ∞ \) as \( Σ_θ \) \( t → 0 \). Thus \( T_k(t)x → 0 \) as \( Σ_θ \) \( t → 0 \), i.e., \( S_k(t) \) is strongly continuous on \( Σ_θ \) \( t → 0 \) and therefore the claim follows from Lemma 4.1.

Case \( k = n \). Since \( \lim_{t→0} C(t)x/t^n = x/n! \) for \( x ∈ X \), we have (cf. (7))

\[ T_n(t)x - x = (-1)^n(πt)^{-1/2} \int_0^∞ [D^k_s \exp(-s^2/4t)] s^n(C(s)x/s^n - x/n!)ds \rightarrow 0 \]
as $\Sigma_\theta \ni t \to 0$ for $x \in X$ and therefore the claim also follows from Lemma 4.1.

In the case $n = 1$, Cor.4.2 (with $d = 0$) and Th.4.3 (with $d = 1$) were shown in [1, Th.5.2]. In the case $d = 0$ we obtain

**Corollary 4.4.** Let $(A, C(t)) \in G_0(X)$. Then for every $\theta \in (0, \pi/2)$

(a) If $n = 2m$ then $(A, S(t)) \in H_m(\theta, \omega^2/\cos^2 \theta, X)$, where $S(t) = T_0(t)$ for $t \in \Sigma_\theta$ and $S(0) = \delta_{0m} I$.

(b) If $n = 2m + 1$ then $(A, T_1(t)) \in \tilde{G}(X)$ and $(A, S(t)) \in H_{m+1}(\theta, \omega^2/\cos^2 \theta, X)$, where $S(t) = T_{-1}(t)$ for $t \in \Sigma_\theta$ and $S(0) = 0$.

The results relating to Cor.4.4 see [5, 8].

5. INTERPOLATION AND EXTRAPOLATION

In this section we will give some results on interpolation and extrapolation of integrated cosine functions, which are analogous to the results of integrated semigroups obtained by Arendt et al [2]. In the sequel, $X$, $Y$ and $Z$ are Banach spaces. $X \hookrightarrow Y$ means that $X$ is a subspace of $Y$ and that the inclusion is continuous. We write $X \hookrightarrow_d Y$, if in addition $X$ is dense in $Y$.

**Theorem 5.1.** Let $B \in G_0(Y)$ and $[D(A'^m)] \hookrightarrow X \hookrightarrow Y$ for some $m \in \mathbb{N}$. In the case $m \geq 2$ assume in addition that $R(\lambda_0, B)X \subset X$ for some $\lambda_0 \in \rho(B)$. Then $B_X \in G_{2m}(X)$.

**Proof.** Let $(B, S(t)) \in G_0(\omega, Y)$ and $C(t) = \int_0^t (t - s)^{2m-1} S(s)ds/(2m - 1)!$. Since by Prop.1.2(f) (with $n = 0$)

$$C(t) = \sum_{i=0}^{m} \binom{m}{i} \lambda^{m-i} (-1)^i \left\{ R(\lambda, B)^m \int_0^t (t - s)^{2m-2i-1} S(s)ds + \sum_{j=m-1}^{m-1} \binom{t^{2j}}{(2j)!} B^{j-m+1} R(\lambda, B)^m \right\}$$

for $t \geq 0$, our assumptions (noting $R(\lambda_0, B)X \subset R(\lambda_0, B)Y \subset D(B) \subset X$ for $m = 1$) imply the strongly continuity of $C(t)X$ and $\|C(t)X\|_X \leq Me^{\omega t}$. The claim now follows from

$$\lambda R(\lambda^2, B_X) = \lambda R(\lambda^2, B) = \int_0^\infty e^{-\lambda x} S(s)xds = \lambda^{2m} \int_0^\infty e^{-\lambda x} C(s)xds$$

for $\lambda > \omega$.

Similarly to [1, Lemma 2.4] we can show that if $(A, C(t)) \in G_{2m}(Y)$, and if $D(A) \neq Y$ in the case $m = 0$ and $C(\cdot)y \notin C(R^+, [D(A)])$ for some $y \in Y$ in the case $m > 0$, then $[D(A)] \hookrightarrow X \hookrightarrow Y$ for some $X$, $(A_X, S(t)) \in G_{2m+2}(X)$, and $S(\cdot)x \notin [D(A_X)] \hookrightarrow X$ for some $X$. From this we can deduce the following theorem (cf. the proof of [1, Th.0.1(b)]).

**Theorem 5.2.** Let $B \in G_0(Y)$ and $(D(B)) \neq Y$. Then $[D(B^m)] \hookrightarrow X \hookrightarrow Y$ for some $X$, and $B_X \in G_{2m}(X)$.

The converse of Th.5.1 and 5.2, i.e., an extrapolation of integrated cosine functions is as follows.

**Theorem 5.3.** Let $A \in G_{2m}(\omega, X)$. Then $[D(A^m)] \hookrightarrow X \hookrightarrow Y$ for some $Y$, $A = B_X$ for some $B \in G_0(Y)$, and $[D(A^m)]$ is maximal unique (i.e., $Z \hookrightarrow X$ and $A_Z \in G_0(\omega, Z)$ imply $Z \hookrightarrow [D(B^m)]$).

**Proof.** Let $C(t)$ be generated by $A$. We first show that for $x \in D(A^m)$ and $t, s \geq 0$

$$2C(2m)(t)C(2m)(s)x = C(2m)(t + s)x + C(2m)(|t - s|)x.$$  \hfill (8)

In fact, set $P_\lambda = \lambda R(\lambda^2, A)$ for $\lambda > \omega$ and $y = R(\lambda_0, A)^m x$ for some $\lambda_0 > \omega$, then

$$\int_0^\infty \int_0^\infty 2e^{-\lambda \mu}C(2m)(t)C(2m)(s)ysdtds = 2P_\lambda P_\mu y = \frac{1}{\lambda - \mu}(P_\lambda - P_\mu)y + \frac{1}{\lambda + \mu}(P_\lambda + P_\mu)y$$

$$\int_0^\infty \int_0^\infty e^{-\lambda \mu}C(2m)(t + s)y + C(2m)(|t - s|)ysdtds$$

and so (8) holds for $y$. Integrating this 2m-times with respect to $t$, we find that the equation obtained holds for $x$. Now, (8) follows by differentiating this equation 2m-times with respect to $t$.

Next, let $Y$ be the completion of $X$ with respect to $\|x\|_Y \equiv \sup_{t \geq 0} \|e^{\omega t}C(2m)(t)R(\lambda_0, A)^m x\|_X$. Then $X \hookrightarrow Y$. Since we can show by (8) and Prop.1.2 that

$$\int_0^\infty (-s)^k e^{-\lambda \mu}C(2m)(t + s)x + C(2m)(|t - s|)x ds = C(2m)(t)(\lambda R(\lambda^2, A))^{(k)} x$$
for \( x \in D(\Lambda^m) \), \( k \in \mathbb{N}_0 \) and \( \lambda > \omega \), it follows that
\[
\| (\lambda R(\lambda^2, A))^{(k)} x \|_Y \leq \frac{1}{k!} \sup_{t \geq 0} \left\{ \int_0^t s^k e^{(\omega - \lambda)s} ds + \int_0^t s^{k-1} e^{(\omega - \lambda)s-2\omega t} ds + \int_0^t s^k e^{-(\omega + \lambda)s} ds \right\} \| x \|_Y
\]
\[
\leq k! (\lambda - \omega)^{-k} \| x \|_Y \quad \text{for} \quad \lambda > \omega, \ k \in \mathbb{N}_0 \ \text{and} \ x \in X. \tag{9}
\]
It follows that \( R(\lambda^2, A) \) has a unique extension \( R(\lambda^2) \in L(Y) \) and so \( R(\lambda^2) (\lambda > \omega) \) is a pseudo resolvent on \( Y \). We show that \( \lim_{\lambda \to \infty} \| \lambda R(\lambda)x - y \|_Y = 0 \) for \( y \in Y \). Indeed, from (9) (with \( k = 0 \)) it suffices to show this for \( y \in X \). Let \( x = R(\lambda, A)^m y \). Then the same method as in the proof of [1, (3.5)] leads to
\[
\| \lambda^2 R(\lambda^2)y - y \|_Y = \sup_{t > 0} \| e^{-\omega t}(\lambda^2 R(\lambda^2, A) - 1)C^{(2m)}(t)\|_X \leq I_+ + I_- \to 0
\]
as \( \lambda \to 0 \) where \( I_+ = \frac{1}{2} \sup_{t > 0} \int_0^t \| e^{-\lambda(C^{(2m)}(|s \pm t|) - C^{(2m)}(|t - s|))} xds\|_X \). Thus there exists a densely defined linear operator \( B \) on \( Y \) such that \( (\omega^2, \infty) \subset \rho(B) \) and \( R(\lambda) = R(\lambda, B) \) for \( \lambda > \omega^2 \). It now follows from (9) that \( B \in G_0(Y) \). The rest can be proved as in [2, §3].

COROLLARY 5.4 If \( D(A) = X \) and \( \rho(A) \neq 0 \). Then the following statements are equivalent.
(a) \( A \in G_{2m}(X) \).
(b) \( [D(A^m)] \hookrightarrow Z \hookrightarrow X \) for some \( Z \), and \( A_Z \in G_0(Z) \).
(c) \( [D(B^m)] \hookrightarrow X \hookrightarrow Y \) for some \( Y \) and some \( B \in G_0(Y) \), \( R(\lambda_0, B)X \subset X \) for some \( \lambda_0 \in \rho(B) \), and \( A = H_X \).

After the paper was accepted, the author understood that the equivalence of (a) and (b) has been extended to the case \( \overline{D(A)} \neq X \) by Shaw and Li [7]. But Th.2.1 and 2.2 cannot be deduced by combining this with the corresponding results of cosine functions because the norm \( \| \cdot \|_m \) is not equivalent to the norm of \( Z \) (see Cor.5.4(b)).

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