THE REGULAR OPEN-OPEN TOPOLOGY FOR FUNCTION SPACES

KATHRYN F. PORTER
Department of Mathematical Sciences
Saint Mary's College of California
Moraga, CA 94575

(Received July 6, 1993 and in revised form March 13, 1995)

ABSTRACT. The regular open-open topology, $T_{r^o}$, is introduced, its properties for spaces of continuous functions are discussed, and $T_{r^o}$ is compared to $T_o$, the open-open topology. It is then shown that $T_{r^o}$ on $H(X)$, the collection of all self-homeomorphisms on a topological space, $(X, T)$, is equivalent to the topology induced on $H(X)$ by a specific quasi-uniformity on $X$, when $X$ is a semi-regular space.

KEY WORDS AND PHRASES. Compact-open topology, admissible topology, open-open topology quasi-uniformity, regular open set, semi-regular space, topological group.

1992 AMS SUBJECT CLASSIFICATION CODES. Primary 54C35, 57S05 Secondary 54H99.

1. INTRODUCTION.

A set-set topology is one which is defined as follows: Let $(X, T)$ and $(Y, T')$ be topological spaces. Let $U$ and $V$ be collections of subsets of $X$ and $Y$, respectively. Let $F \subseteq Y^X$, the collection of all functions from $X$ into $Y$. Define, for $U \in U$ and $V \in V$, $(U, V) \in \{f \in F : f(U) \subseteq V\}$. Let $S(U, V) = \{(U, V) : U \in U$ and $V \in V\}$. If $S(U, V)$ is a subbasis for a topology $T(U, V)$ on $F$ then $T(U, V)$ is called a set-set topology.

Some of the most commonly discussed set-set topologies are the compact-open topology, $T_{co}$, which was introduced in 1945 by R. Fox [1], and the point-open topology, $T_p$. For $T_{co}$, $U$ is the collection of all compact subsets of $X$ and $V = T^*$, the collection of all open subsets of $Y$, while for $T_p$, $U$ is the collection of all singletons in $X$ and $V = T^*$.

In section 2 of this paper, we shall introduce and discuss the regular open-open topology for function spaces. It will be shown which of the desirable properties $T_{r^o}$ possesses. In section 3, Pervin and almost-Pervin spaces are explained.

The fact that $T_{r^o}$, on $H(X)$, is actually equivalent to the regular-Pervin topology of quasi-uniform convergence will be discussed in section 4 along with the topic of quasi-uniform convergence. The advantage of the regular open-open topology is the set-set notation which provides us with
simple notation and, hence, our proofs are more concise than those using the cumbersome notation of the quasi-uniformity.

We assume a basic knowledge of quasi-uniform spaces. An introduction to quasi-uniform spaces may be found in Fletcher and Lindgren’s [2] or in Murdeshwar and Naimpally’s book [3]. Throughout this paper we shall assume \((X, T)\) and \((Y, T^*)\) are topological spaces.

2. THE REGULAR OPEN-OPEN TOPOLOGY.

A subset, \(W\), of \(X\) is called a regular open set provided \(W \cap \text{Int}(\text{Cl}(W))\). If we let \(U\) be the collection of all regular open sets in \(X\) and \(V = T^*\), then \(S_{r oo} = S(U, V)\) is the subbasis for a topology, \(T_{r oo}\), on any \(F \subseteq Y^X\), which is called the regular open-open topology.

A topological space, \(X\), is called semi-regular provided that for each \(U \subseteq X\) and each \(x \in U\) there exists a regular open set, \(V\), in \(X\), such that \(x \in V \subseteq U\). One can easily show that if \((X, T)\) is a semi-regular space then \(T_{r oo} \subseteq T_{oo}\), the open-open topology (Porter, [4]) which has as a subbasis the set \(S_{oo} = \{(U, V) : U \in T \text{ and } V \in T^*\}\).

We now examine some of the properties of function spaces the regular open-open topology possesses. The first two theorems also hold for the open-open topology even when \(X\) is not semi-regular. The proofs of these two theorems are straightforward and are left to the reader.

THEOREM 1. Let \((X, T)\) be a semi-regular space and \(F \subseteq C(X, Y)\). If \((Y, T^*)\) is Ti for 0, 1, 2, then \((F, T_{r oo})\) is Ti for 0, 1, 2.

A topology, \(T^*\), on \(F \subseteq Y^X\) is called an admissible (Arens [5]) topology for \(F\) provided the evaluation map, \(E: (F, T^*) \times (X, T) \rightarrow (Y, T^*)\), defined by \(E(f, x) = f(x)\), is continuous.

THEOREM 2. If \(F \subseteq C(X, Y)\) and \(X\) is semi-regular, then \(T_{r oo}\) is admissible for \(F\).

Arens also has shown that if \(T^*\) is admissible for \(F \subseteq C(X, Y)\), then \(T^*\) is finer than \(T_{oo}\). From this fact and Theorem 2, it follows, as it does for \(T_{oo}\), that \(T_{oo} \subseteq T_{r oo}\) when \(X\) is semi-regular.

THEOREM 3. The sets of the form \((U, V)\) where both \(U\) and \(V\) are regular open sets in \(X\) form a subbasis for \((H(X), T_{r oo})\).

PROOF. Let \((U, V)\) be a subbasic open set in \((H(X), T_{r oo})\). i.e., \(U\) is a regular open set and \(O\) is an open set, not necessarily regular. Let \(f \in (U, O)\). Then \(f(U) \subseteq O\), so \(f \in (U, f(U)) \subseteq (U, O)\) and \(f(U)\) is a regular open set.

Let \((G, o)\) be a group such that \((G, T)\) is a topological space, then \((G, T)\) is a topological group provided the following two maps are continuous. (1) \(m: G \times G \rightarrow G\) defined by \(m(g_1, g_2) = g_1 \circ g_2\) and \(\Phi: G \rightarrow G\) defined by \(\Phi(g) = g^{-1}\). If only the first map is continuous, then we call \((G, T)\) a quasi-topological group (Murdeshwar and Naimpally [3]).

Note that \(H(X)\) with the binary operation \(o\), composition of functions, and identity element \(e\), is a group. It is not difficult to show that if \((X, T)\) is a topological space and \(G\) is a subgroup of \(H(X)\) then \((G, T_{oo})\) is a quasi-topological group. However, \((G, T_{oo})\) is not always a topological group (Porter, [4]) since \(\Phi\) is not always continuous although \(m\) is always continuous. But we discover the following about the regular open-open topology.
THEOREM 4. Let $X$ be a semi-regular space and let $G$ be a subgroup of $H(X)$. Then $(G, T_{\text{too}})$ is a topological group.

PROOF. Let $X$ be a semi-regular space and let $G$ be a subgroup of $H(X)$. Let $(U, V)$ be a subbasic open set in $T_{\text{too}}$ such that both $U$ and $V$ are regular open sets. Let $(f, g) \in m^{-1}((U, V))$. Then, $f \circ g(U) \subset V$ and $g(U) \subset f^{-1}(V)$. So, $(f, g) \in (g(U), V) \times (U, g(U)) \in T_{\text{too}} \times T_{\text{too}}$. But $(g(U), V) \times (U, g(U)) \subset m^{-1}((U, V))$. Thus, $m$ is continuous.

Note that the inverse map $\Phi : G \to G$ is bijective and that $\Phi^{-1} = \Phi$. Thus, in order to show that $\Phi$ is continuous, it suffices to show that $\Phi$ is an open map. To this end, let $(O, U)$ be a subbasic open set in $T_{\text{too}}$ where $O$ and $U$ are both regular open sets. Clearly, $\Phi((O, U)) = ((X \setminus U), (X \setminus O))$ since we are dealing with homeomorphisms. Note that if $C, K$ are regular closed sets then $\text{Int}_X C, \text{Int}_X K$ are regular open sets. Thus, since $(X \setminus O), (X \setminus U)$ are regular closed sets, $\text{Int}_X(X \setminus U), \text{Int}_X(X \setminus O)$ are regular open sets. Again, since $G$ is a set of homeomorphisms, $(X \setminus U, X \setminus O) = (\text{Int}_X(X \setminus U), \text{Int}_X(X \setminus O))$ but this is in $T_{\text{too}}$. Therefore, $\Phi(O, U)$ is an open set in $T_{\text{too}}$. So, $\Phi$ is open and we are done.

3. PERVIN AND ALMOST-PERVIN SPACES.

A topological space, $(X, T)$, is called a Pervin space (Fletcher [4]) provided that for each finite collection, $A$, of open sets in $X$, there exists some $h \in H(X)$ such that $h \neq e$ and $h(U) \subset U$ for all $U \in A$. A topological space, $(X, T)$, is called almost-Pervin provided that for each finite collection, $A$, of regular open sets, there exists some $h \in H(X)$ such that $h \neq e$ and $h(O) \subset O$ for all $O \in A$.

Topologies are rarely interesting if they are the trivial or discrete topology. We have previously shown (Porter, [4]) that $(H(X), T_{\text{too}})$ is not discrete if and only if $(X, T)$ is a Pervin space. The situation for $T_{\text{too}}$ is similar.

THEOREM 5. $(H(X), T_{\text{too}})$ is not discrete if and only if $(X, T)$ is almost-Pervin.

PROOF. First, assume that $(X, T)$ is an almost-Pervin space. Let $W$ be a basic open set in $T_{\text{too}}$ which contains $e$; i.e. $W = \bigcap_{i=1}^{n} (O_i, U_i)$ where $O_i \subset U_i$ for each $i = 1, 2, 3, \ldots, n$ and $O_i$ and $U_i$ are regular open sets in $X$. The collection $\{O_i : i = 1, 2, 3, \ldots, n\}$ is a finite collection of regular open sets in $X$, and $X$ is an almost-Pervin space, hence, there exists some $h \in H(X)$ such that $h \neq e$ and $h(O_i) \subset O_i \subset U_i$. So, $h \in W$ and $h \neq e$. Therefore, $(H(X), T_{\text{too}})$ is not a discrete space.

Now assume that $(H(X), T_{\text{too}})$ is not discrete. Let $V$ be a finite collection of regular open sets in $X$. Let $O = \bigcap_{U \in V} (U, U)$. Then, $O$ is a basic open set in $(H(X), T_{\text{too}})$ which is not a discrete space. Hence, there exists $h \in O$ with $h \neq e$. So, $(X, T)$ is almost-Pervin.

The above proof, along with the few needed definitions involving $T_{\text{too}}$, is an example of the simplification that the definition of $T_{\text{too}}$ offers over the quasi-uniform definition and notation.

4. QUASI-UNIFORM CONVERGENCE.

Recall that if $Q$ is a quasi-uniformity on $X$, then the topology, $T_Q$, on $X$, which has as its
neighborhood base at \( x \), \( B_x = \{ U[x] : U \in Q \} \), is called the topology induced by \( Q \). The ordered triple \( (X, Q, T_Q) \) is called a quasi-uniform space. A topological space, \( (X, T) \) is quasi-uniformizable provided there exists a quasi-uniformity, \( Q \), such that \( T_Q = T \). In 1962, Pervin [7] proved that every topological space is quasi-uniformizable by giving the following construction.

Let \( (X, T) \) be a topological space. For each \( O \in T \), define the set \( S_O = (O \times O) \cup ((X \setminus O) \times X) \). Let \( S = \{ S_O : O \in T \} \). Then \( S \) is a subbasis for a quasi-uniformity, \( P \), for \( X \), called the Pervin quasi-uniformity and, as is easily shown, \( T_P = T \).

If we use the same basic structure as above but change the subbasis to \( S = \{ S_O : O \) is a regular open set \} then the quasi-uniformity induced will be called the regular-Pervin quasi-uniformity, \( RP \).

If \( (X, Q) \) is a quasi-uniform space then \( Q \) induces a topology on \( H(X) \) called the topology of quasi-uniform convergence w.r.t. \( Q \), as follows: For each set \( U \in Q \), let us define \( W(U) = \{ (f, g) \in H(X) \times H(X) : (f(z), g(x)) \in U \) for all \( z \in X \} \). Then, \( B(Q) = \{ W(U) : U \in Q \} \) is a basis for \( Q^* \), the quasi-uniformity of quasi-uniform convergence w.r.t. \( Q \) (Naimpally [8]). Let \( T_{Q^*} \) denote the topology on \( H(X) \) induced by \( Q^* \). \( T_{Q^*} \) is called the topology of quasi-uniform convergence w.r.t. \( Q^* \).

If \( P \) is the Pervin quasi-uniformity on \( X \), \( T_P \) is the Pervin topology of quasi-uniform convergence and if \( RP \) is the regular-Pervin quasi-uniformity on \( X \), then \( T_{RP} \) is called the regular-Pervin topology of quasi-uniform convergence, \( T_{RP^*} \).

It has been shown that the open-open topology is equivalent to the Pervin topology of quasi-uniform convergence (Porter, [4]). It is also true that the regular open-open topology is equivalent to the regular-Pervin topology of quasi-uniform convergence. The method of two proofs are exactly the same and we leave this one for the reader.

THEOREM 6. Let \( (X, T) \) be a topological space and let \( G \) be a subgroup of \( H(X) \). Then, \( T_{oo} = T_{RP^*} \) on \( G \).

ACKNOWLEDGEMENT. The author would like to thank the Committee for Faculty and Curriculum Development at Saint Mary's College for their financial support.

REFERENCES