ON APPROXIMATION OF FUNCTIONS AND THEIR DERIVATIVES
BY QUASI-HERMITE INTERPOLATION

G. MIN
CECM, Department of Mathematics and Statistics
Simon Fraser University
Burnaby, B.C., Canada V5A 1S6

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ABSTRACT. In this paper, we consider the simultaneous approximation of the derivatives of the
functions by the corresponding derivatives of quasi-Hermite interpolation based on the zeros of \((1 - x^2)p_n(x)\) (where \(p_n(x)\) is a Legendre polynomial). The corresponding approximation degrees are given.
It is shown that this matrix of nodes is almost optimal.

KEY WORDS: Hermite interpolation, optimal nodes, derivatives, Legendre polynomials, best approximation.

1 INTRODUCTION.

Let

\[-1 \leq x_n < \ldots < x_1 < x_0 \leq 1\]  \hspace{1cm} (1.1)

be an arbitrary nodes system on \([-1,1]\) and let \(f \in C^4[-1,1]\). We consider the Hermite interpolation operator:

\[H_n(f,x) := \sum_{k=0}^n f(x_k)h_k(x) + \sum_{k=0}^n f'(x_k)\sigma_k(x),\]

(1.2)

where

\[h_k(x) = v_k(x)l_k^2(x), \quad \sigma_k(x) = (x-x_k)l_k^2(x),\]

\[l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x-x_k)},\]

\[v_k(x) = 1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x-x_k), \quad \omega(x) = \Pi_{k=0}^n(x-x_k).\]

It satisfies the following conditions:

\[H_n(f,x_k) = f(x_k), \quad k = 0,1,\ldots,n\]

and

\[H'_n(f,x_k) = f'(x_k), \quad k = 0,1,\ldots,n\]

There have been many articles considering the problem of approximation to \(f(x)\) by \(H_n(f,x)\). Generally, we consider approximation of \(f'(x)\) by the derivative of Hermite interpolation. We know that the convergence

\[\lim_{n \to \infty} \|H'_n(f,x) - f'(x)\| = 0,\]

does not hold for all \(f \in C^4[-1,1]\) (here \(\|\cdot\|\) is the maximum norm). Pottinger [1] investigated this problem when \(\{x_k\}_{k=0}^n\) are the zeros of the Tchebycheff polynomial of the first kind and obtained the following result:
\[ ||H_n^r(f, x) - f'(x)|| = O(n)E_{2n}(f'), \quad (1.3) \]

where \( E_n(f) \) is the best approximation of \( f(x) \). (The factor \( O(n) \) is best possible, cf. Steinhaus [2].) In [3], Szabados and Varma introduced a norm for the higher derivatives of the operator (1.2):

\[ ||H_n^r|| = \sup(||H_n^r(f, x)|| : |f^{(i)}(x_k)| \leq n'(1 - x_k^2)^{r-i/2}, k = 1, ..., n; i = 0, 1 \]

\((r, n = 1, 2, ...)\) and they proved that for any system of nodes ([3, Theorem 1])

\[ ||H_n^r|| \geq c_r n^r \ln n, \quad (n, r = 1, 2, ...) \]

(1.4)

where \( c_r > 0 \) depends only on \( r \). Moreover, for the matrix of nodes:

\[ \omega(x) = P_{n-2t+1}^{(\alpha, \alpha)}(x)H_{j=1}^t(x^2 - \cos^2 \left( \frac{(j-1)\pi}{3t(n-2t+1)} \right), \]

\( (1.5) \)

they obtain ([3, Theorem 3])

\[ ||H_n^r|| = O(n^r \ln n), \quad (1.6) \]

where \( t = \lfloor \frac{r+3}{4} \rfloor, \alpha = 2t - \frac{r+1}{2} (r \geq 1 \text{ integer}) \) and \( P_{n-2t+1}^{(\alpha, \alpha)}(x) \) are the ultraspherical Jacobi polynomials of degree \( n - 2t \). Moreover, \( \alpha \) takes only the values \(-1/2, 0, 1/2, 1\) according to \( r \equiv 0, 3, 2, 1 (\text{mod } 4) \). (See [3, Remark, P305].) Therefore for the matrix of nodes defined by (1.5) we have

\[ ||H_n^r(f, x) - f'(x)|| = O(\ln n)\omega(f'), \quad (1.7) \]

(see [3]) At the end of paper [3], they speculated that "it would be interesting to construct a matrix which is optimal for all the derivatives up to order \( r \)." This is the problem of constructing matrix nodes so that the corresponding simultaneous approximation of \( f(x) \) from the first derivative to the \( r \)-th derivative is optimal by the corresponding Hermite interpolation.

Remark: With respect to Lagrange interpolation, the complete solution of minimizing the corresponding derivatives norm to (1.4) was given by Szabados [4] (also see Vértesi [5]). The main idea is that adding nodes (near \( x_1 \)) to Jacobi nodes make the similar estimates of (1.4) optimal.

In this paper, we point out that for the quasi-Hermite interpolation \( R_n(f, x) \) based on the zeros of \((1 - x^2)p_n(x)\) (where \( p_n(x) \) is the Legendre polynomial with normalization: \( p_n(1) = 1 \)), we have

**THEOREM 1.** If \( f \in C^1[-1, 1] \), then

\[ ||R_n^r(f, x) - f'(x)|| = O(\ln n)E_{2n}(f'). \]

**THEOREM 2.** If \( f \in C^r[-1, 1] \) \((r \geq 2)\), then

\[ ||R_n^r(f, x) - f'(x)|| = O(\ln n)E_{2n}(f') = O\left( \frac{\ln n}{n} \right)E_{2n-1}(f''), \]

\( (1.8) \)

\[ ||\sqrt{1 - x^2}(R_n^r(f, x) - f''(x))|| = O(\ln n)E_{2n-1}(f'''), \]

\( (1.9) \)

and

\[ ||R_n^r(f, x) - f^{(i)}(x)|| = O(\ln n)E_{2n-i+1}(f^{(i)}), \quad i = 2, ..., r \]

\( (1.10) \)

where \( 0 < \sigma < 1 \).

From this we see that the zeros of \((1 - x^2)p_n(x)\) are almost optimal and the corresponding simultaneous approximation is better than that of Hermite interpolation based on the zeros of Tchebysheff polynomial of the first kind.

Remark: We conjecture that the factor \( \sqrt{1 - x^2} \) in (1.10) cannot be removed on the whole interval \([-1, 1]\), in which case the preceding results are optimal.
2 LEMMAS.

In order to prove the Theorems, we state some properties of Legendre polynomials (see Szegő [6]).

\[ |p_n(x)| \leq 1, \quad (2.1) \]
\[ (1 - x^2)^{1/4}|p_n(x)| \leq (2\pi n)^{-1/2}, \quad n \geq 2 \quad (2.2) \]
\[ (1 - x^2)^{3/4}|p'_n(x)| \leq (2n)^{1/2}, \quad n \geq 3 \quad (2.3) \]
\[ \sin^2 \theta_k = 1 - x_k^2 > (k - 3/2)^2n^{-2}, \quad k = 1, \ldots, \lfloor n/2 \rfloor \quad (2.4) \]
\[ |p'_n(x_k)| > c(k - 3/2)^{-3/2}n^2, \quad k = 1, \ldots, \lfloor n/2 \rfloor \quad (2.5) \]

We note that in (2.4) and (2.5) similar estimates are hold for \( k = \lfloor n/2 \rfloor, \ldots, n \). On combining (2.4) and (2.5), it follows that

\[ [(1 - x_k^2)^{3/4}|p'_n(x_k)|]^2 \geq cn, \quad k = 1, \ldots, n \quad (2.6) \]

where \( c \) is an absolute positive constant independent of \( f \) and \( n \), whose value may vary from line to line through our paper.

Let

\[ -1 = x_{n+1} < x_n < \ldots < x_1 < x_0 = 1 \]

be the zeros of \((1 - x^2)p_n(x)\). Then its corresponding quasi-Hermite interpolation is the following

\[ R_n(f, x) = \sum_{k=0}^{n+1} f(x_k)r_k(x) + \sum_{k=1}^{n} f'(x_k)\gamma_k(z), \quad (2.7) \]

where

\[ r_0(x) = \frac{1 + x}{2} p_n(x), \quad r_{n+1} = \frac{1 - x}{2} p_n(x), \]
\[ r_k(z) = \frac{1 - x_k^2}{1 - x_k^2} l_k(z), \quad k = 1, \ldots, n \]
\[ \gamma_k(z) = (z - x_k)r_k(z), \quad k = 1, \ldots, n \]
\[ l_k(z) = \frac{p_n(z)}{r_n(x_k)(z - x_k)}, \quad k = 1, \ldots, n \]

It satisfies that

\[ R_n(f, x_k) = f(x_k), \quad k = 0, 1, \ldots, n + 1. \]

and

\[ R'_n(f, x_k) = f'(x_k), \quad k = 1, \ldots, n \]

**LEMMA 1.** We have

\[ \sqrt{1 - x_k^2} \leq \sqrt{1 - x^2} + 2\frac{|x - x_k|}{\sqrt{1 - x_k^2}}, \quad k = 1, \ldots, n. \]

**PROOF.** One easily sees that

\[ \sqrt{1 - x_k^2} = \sqrt{1 - x^2} + \frac{x^2 - x_k^2}{\sqrt{1 - x^2} + \sqrt{1 - x_k^2}} \leq \sqrt{1 - x^2} + 2\frac{|x - x_k|}{\sqrt{1 - x_k^2}}. \]

This proves Lemma 1. \( \Box \)
LEMMA 2. We have

(i) \[ I_1 := \sum_{k=1}^{n} \frac{|x - x_k|}{1 - x_k^2} l_k^2(x) = O(\ln n) \quad (2.8) \]

(ii) \[ I_2 := \sum_{k=1}^{n} |x - x_k| \frac{1 - x^2}{1 - x_k^2} |l_k(x)l'_k(x)| = O(\ln n) \quad (2.9) \]

PROOF. From Lemma 1 we have

\[ I_1 \leq \sum_{k=1}^{n} \frac{\sqrt{1 - x^2}|x - x_k|}{(1 - x_k^2)^{3/2}} l_k^2(x) + 2 \sum_{k=1}^{n} \frac{|x - x_k|^2}{(1 - x_k^2)^{3/2}} l_k^2(x) := A_1(x) + A_2(x) \quad (2.10) \]

Throughout this paper we assume \( x \) to be the zero of \( p_n(x) \) which is the nearest to \( x \) and \( i = |k - j| \).

By using (5.8) in Prasad and Varma\[7\] we have

\[ \sqrt{1 - x^2} \frac{|x - x_j|}{1 - x_j^2} l_j^2(x) \leq \frac{c}{n}. \quad (2.11) \]

Notice that, with \( x = \cos \theta \) \((0 \leq \theta \leq \pi)\)

\[ \sin \theta \leq \sin \theta + \sin \theta_k \leq 2 \sin \frac{\theta + \theta_k}{2}, \]

so we have

\[ A_1(x) = \frac{1}{\sqrt{1 - x_j^2}} \left\{ \frac{\sqrt{1 - x^2}|x - x_j|}{1 - x_j^2} l_j^2(x) + \sum_{i \neq j} \frac{\sqrt{1 - x^2}|x - x_j|}{(1 - x_j^2)^{3/2}} l_i^2(x) \right\} \]

\[ \leq \frac{c}{n \sin \theta_j} + \sum_{i \neq j} \left( \frac{1}{(1 - x_j^2)^{3/4}} \right) |p_n(x_k)|^2 |x - x_k| \]

\[ = O(1)\left[ 1 + p_n^2(x) \sum_{i \neq j} \frac{1}{\sin |x - x_k|} \right] = O(1)\left[ 1 + \frac{p_n^2(x)}{n} \sum_{i \neq j} \right] = O(\ln n). \]

Similarly,

\[ A_2(x) = \sum_{k=1}^{n} \frac{p_n^2(x)}{(1 - x_k^2)^{3/4}|p_n(x_k)|^2} \frac{1}{\sqrt{1 - x_k^2}} = O(1) \frac{p_n^2(x)}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{1 - x_k^2}} = O(\ln n), \]

so we obtain (2.8).

Notice that

\[ l'_k(x) = \frac{p_n(x)(x - x_k) - p_n(x)}{(x - x_k)^2 p_n'(x_k)}, \]

and we have

\[ I_2 \leq \sum_{k=1}^{n} |x - x_k| \frac{(1 - x^2)|x - x_k| |p'_n(x_k)|}{(1 - x_k^2)^{3/4}|p_n(x_k)|^2} |l_k(x)| + \sum_{k=1}^{n} r_k(x) := B_1(x) + B_2(x) \]

One notes Prasad and Varma\[7\]

\[ \frac{(1 - x^2)^{1/4}}{(1 - x_k^2)^{1/4}} |l_k(x)| \leq c, \]

so we have

\[ B_1(x) = \sum_{k=1}^{n} \frac{(1 - x^2)^{3/4}|p'_n(x_k)|}{(1 - x_k^2)^{3/4}|p_n(x_k)|} \frac{(1 - x^2)^{1/4}}{(1 - x_k^2)^{1/4}} |l_k(x)| \]

\[ = O(1) \left\{ \frac{(1 - x^2)^{3/4}|p'_n(x_k)|}{(1 - x_k^2)^{3/4}|p_n(x_k)|} \right\} \left\{ \frac{(1 - x^2)(1 - x_k^2)^{1/4}}{(1 - x_k^2)^{3/4}|p_n(x_k)|^2} |x - x_k| \right\} \]

\[ = O(1) \left[ 1 + \frac{(1 - x^2)|p_n(x)|p'_n(x)}{n} \sum_{i \neq j} \frac{\sin \theta_k}{|x - x_k|} \right] \]

\[ = O(1)\left[ 1 + \ln n (1 - x^2)|p_n(x)|p'_n(x) \right] = O(\ln n). \]
Obviously,

\[ B_2(x) \leq \sum_{k=0}^{n+1} r_k(x) = 1. \]

Therefore we obtain (2.9). \qed

**Lemma 3.** We have

\[ I_3 := \sum_{k=0}^{n+1} (1 - x_k^2) |r_k(x)| = O(\ln n)(1 - x^2), \tag{2.12} \]

and

\[ I_4 := \sum_{k=1}^{n} \sqrt{1 - x_k^2} |\gamma_k(x)| = O\left(\frac{\ln n}{n}\right)\sqrt{1 - x^2}. \tag{2.13} \]

**Proof.** Since

\[ I_3 = (1 - x^2) \sum_{k=1}^{n} I_k^2(x), \]

from Nevai and Vértesi [8] we have

\[ \sum_{k=1}^{n} I_k^2(x) = O(1)(1 + \frac{J_n^2(x)}{n} + \frac{\ln n}{n} J_n^2(x)), \]

where \( J_n(x) \) is the orthonormal Legendre polynomials:

\[ \int_{-1}^{1} J_n(x) J_m(x) \, dx = \delta_{nm}, \]

and notice that Natanson [9] gives

\[ ||J_n(x)|| = O(1)n^{1/2}. \]

It follows that

\[ \sum_{k=1}^{n} I_k^2(x) = O(\ln n), \]

this implies (2.12). Also, we have

\[ I_4 = \sum_{k=1}^{n} \frac{(1 - x^2)|x - x_k| I_k^2(x)}{\sqrt{1 - x_k^2}} \]

\[ = (1 - x^2) \frac{(1 - x_k^2)^{1/4} p_n(x_k)}{(1 - x_k^2)^{3/4} p_n'(x_k)} |I_j^2(x)| + \sum_{k \neq j} \frac{(1 - x_k^2) p_n^2(x_k) (1 - x_k^2)}{[(1 - x_k^2)^{3/4} p_n'(x_k)]^2 |x - x_k|}. \]

Recall that (Erdős [10]) for \(-1 \leq x \leq 1, \)

\[ |I_k(x)| \leq 1, \quad k = 1, \ldots, n \]

therefore, similar to the estimates of \( I_1 \) and \( I_2, \) we have

\[ I_4 = O(1) \frac{1 - x^2}{n} + \frac{(1 - x^2) p_n^2(x)}{n} \frac{1}{\sin |\theta_k - \frac{\pi}{2}|} = O\left(\frac{\ln n}{n}\right) \sqrt{1 - x^2}. \]

This proves Lemma 3. \qed

**Remark:** If we need not want to obtain the factor \((1 - x^2), \) we can obtain a better estimate of \( I_3. \)

**Lemma 4.** Let \( f \in C^r[-1, 1], \) then there exist polynomials \( q_n(x) \) of degree \( n \geq 4r + 5 \) such that \((j = 0, 1, \ldots, r)\)

\[ |f^{(j)}(x) - q_n^{(j)}(x)| = O\left(\frac{\sqrt{1 - x^2}}{n}\right)^{r-j} E_{n-r}(f^{(r)}). \tag{2.14} \]
PROOF. From Gopengauz's Theorem [11] we know that there exist polynomials $t_n(x)$ of degree $n \geq 4r + 5$ such that

$$|f(\omega) - t_n(x)| \leq c\left(\frac{\sqrt{1 - x^2}}{n}\right)^{r-j} \omega(f(r), \frac{\sqrt{1 - x^2}}{n})$$

Let $s_n(x)$ be the polynomial of degree $n > r$ such that

$$||f(r) - s_n(r)(x)|| \leq E_{n-r}(f(r)),$$

then we have

$$|f(\omega) - q_n(x)| := |f(\omega) - (s_n(x) + t_n(x))|$$

$$\leq c\left(\frac{\sqrt{1 - x^2}}{n}\right)^{r-j} \omega((f - s_n(r), \frac{1}{n}) = O(1)(\frac{\sqrt{1 - x^2}}{n})^{r-j}|f(r) - s_n(r)||$$

$$= O(1)(\frac{\sqrt{1 - x^2}}{n})^{r-j} E_{n-r}(f(r)).$$

This proves Lemma 4. □

LEMMA 5. Let $s_j(x)$ be a polynomial of degree $\leq n$, and suppose that the inequality

$$\sum_{j=1}^{m} |s_j(x)| = O(1), \quad -1 \leq x \leq 1.$$

holds. Then

$$(1 - x^2)^{n/2} \sum_{j=1}^{m} |s_j^{(i)}(x)| = O(1)n^i,$$

(2.15)

where $m \geq 1$ and $1 \leq i \leq n$.

PROOF. Although Ramm [12, Lemma 1, p.285] only proved the case of $i=1$, (26) can be obtained by using a completely similar method. □

3 PROOFS OF THEOREMS.

PROOF OF THEOREM 1. Notice that

$$R_n(f, x) - f(x) = \sum_{k=0}^{n+1} (f(x_k) - f(x))r_k(x) + \sum_{k=1}^{n} f'(x_k)\gamma_k(x)$$

This implies

$$||R_n|| \leq \sum_{k=0}^{n+1} |x - x_k| |r_k'(x)| + \sum_{k=1}^{n} |\gamma_k(x)||f'||$$

(3.1)

One easily sees that

$$(1 - x)|r_0'(x)| \leq (1 - x)[p^2_n(x) + (1 + x)|p_n(x)p_n'(x)|] = O(1).$$

Similarly we have

$$(1 + x)|r_{n+1}'(x)| = O(1).$$

Notice that

$$r_k'(x) = \frac{2x}{1 - z_k^2} l_k'(x) + \frac{2(1 - x^2)}{1 - z_k^2} j_k'(x)$$

and

$$\gamma_k'(x) = \tau_k(x) + (z - x_k)r_k'(x).$$

From Lemma 2 we have
and also we have
\[ \sum_{k=0}^{n} |x - x_k| r_k'(x) | = O(\ln n) \]  
(3.2)

It now follows that
\[ \| R_n' \| = O(\ln n) \| f' \|. \]
(3.4)

Combining Lemma 4, (3.2) and (3.3), we obtain Theorem 1. □

PROOF OF THEOREM 2. Theorem 1 implies (9). Here we only prove the case \( i = 2 \). The other cases are completely similar. By using Lemma 5 (or see Borwein and Erdelyi [13]) and from Lemma 3 we obtain the following
\[ \sum_{k=0}^{n+1} (1 - x_k^2) |r_k''(x)| = O(n^2 \ln n) \]
(3.5)

and
\[ \sqrt{1 - x^2} \sum_{k=1}^{n} \sqrt{1 - x_k^2} |\gamma_k'(x)| = O(n \ln n) \]
(3.6)

Notice that
\[ R_n''(f, x) - f''(x) = R_n''(f - q_{2n+1}, x) + q_{2n+1}'(x) - f''(x) \]

and
\[ R_n''(f - q_{2n+1}, x) = \sum_{k=0}^{n+1} (f(x_k) - q_{2n+1}(x_k)) r_k''(x) + \sum_{k=1}^{n} (f'(x_k) - q_{2n+1}'(x_k)) \gamma_k''(x). \]

Combining Lemma 4, (3.5) and (3.6), we obtain (1.10). □

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References


