GENERALIZED PERIODIC RINGS

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ABSTRACT. Let R be a ring, and let N and C denote the set of nilpotents and the center of R, respectively. R is called generalized periodic if for every x e R \ (N u C), there exist distinct positive integers m, n of opposite parity such that x^n - x^m \in N \cap C. We prove that a generalized periodic ring always has the set N of nilpotents forming an ideal in R. We also consider some conditions which imply the commutativity of a generalized periodic ring.

KEY WORDS. AND PHRASES Commutativity, periodic ring, generalized periodic ring, center of a ring, commutator ideal.

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1. INTRODUCTION.

Throughout the paper, R will denote a ring, N the set of nilpotents, C the center, J the Jacobson radical, and C(R) the commutator ideal of R. The ring R is called periodic if for every x in R there exist distinct positive integers m, n such that x^m = x^n. An element x of R is called potent if, for some positive integer n > 1, x^n = x. R is called weakly periodic if every element x of R can be written as a sum of a potent element and a nilpotent element. It is well known that a periodic ring is necessarily weakly periodic. Whether a weakly periodic ring is necessarily periodic is apparently not known, except in the presence of other additional hypotheses. We now formally state the definition of a generalized periodic ring.

Definition. A ring R is called generalized periodic if for every x in R, x \in N \cup C, we have

x^n - x^m \in N \cap C, for some positive integers m, n of opposite parity. (1.1)

Or, equivalently,

x^n - x^{nk} \in N \cap C; n, k \in \mathbb{Z}^+; k \text{ odd}; (x \notin N \cup C). (1.1)
We prove that the set of nilpotents in a generalized periodic ring $R$ is always an ideal in $R$. We also consider conditions which imply the commutativity of a generalized periodic ring.

2. STRUCTURE OF GENERALIZED PERIODIC RINGS.

We begin with some basic facts about generalized periodic rings.

Lemma 1. In a generalized periodic ring $R$, we have

(i) $C(R) \subseteq J$;

(ii) $J \subseteq N \cup C$;

(iii) $N \subseteq J$.

PROOF (i). By a well known theorem of Herstein [1], if $R$ is a division ring which satisfies (1.1), then $R$ is commutative. Next, suppose that $R$ is a primitive ring which satisfies (1.1). Since (1.1) is inherited by all subrings of $R$ and by all homomorphic images of $R$, it follows, by Jacobson's Density Theorem, that if $R$ is not a division ring, then some complete matrix ring $D_m$, with $m > 1$, over a division ring $D$ satisfies (1.1). This, however, is false, as can be seen by taking $x = E_{12} + E_{21}$ in $D_m$. Hence, a primitive ring which satisfies (1.1) is necessarily a division ring, and hence is commutative by Herstein's Theorem. Therefore, any semisimple ring which satisfies (1.1) is commutative, which proves (i).

(ii). Suppose $x \in J, x \in N, x \in C$. Then, by (1.1), $x^n - x^m \in N, n \neq m$, and thus for some $q \in Z^+$, $g(\lambda) \in Z[\lambda]$,

$$x^q = x^{q+1}g(x); \quad (g(\lambda) \in Z[\lambda]). \quad (2.1)$$

It is readily verified that $e = [xg(x)]^n$ is an idempotent element in $J$ (since $x \in J$), and hence $e = [xg(x)]^n = 0$. But, by (2.1), $x^q = x^q, e = 0$, and hence $x \in N$, contradiction. This contradiction proves (ii).

(iii). First, we prove that

$$ax \in N \text{ for all } a \in N \text{ and all } x \in R. \quad (2.2)$$

To show this, first note that by (i) and (ii),

$$C(R) \subseteq N \cup C. \quad (2.3)$$

Suppose (2.2) is false, and let $a \in N, x \in R, ax \in N$ (for some $a$ and $x$). Let $\overline{R} = R / C(R)$, and let $\overline{x} = x + C(R)$ be an arbitrary element of $\overline{R}$. Since $\overline{R}$ is commutative, (2.2) implies that $ax$ is nilpotent, and hence $(ax)^r \in C(R)$ for some positive integer $r$. Thus, by (2.3) $(ax)^r \in N$ or $(ax)^r \in C$. Since, by hypothesis, $ax \in N$, therefore

$$(ax)^r \in C \text{ for some positive integer } r.$$

Since $a \in N$, let $a^0 = 0$. Note that, since $(ax)^r \in C$,

$$(ax)^r(ax)^r = a \cdot (ax)^r \cdot (xa)^{-r} x = a^xt_{t_r} \text{ (some } t_r \in R).$$

Continuing this process, we see that

$$[(ax)^r]^k = a^kt_{k-1} \text{ (some } t_{k-1} \in R), \text{ for all } k \in Z^+. \quad \text{In particular,}$$

$$[(ax)^r]^0 = a^xt_{r-1} = 0 \text{ (since } a^0 = 0),$$

and hence $ax \in N$, contradiction. This contradiction proves (2.2). To complete the proof of (iii), let $a \in N, x \in R$. Then, by (2.2), $ax \in N$ and hence $ax$ is right quasi-regular for all $x$ in $R$, which implies $a \in J$. This completes the proof of the lemma.

We are now in a position to prove the following fundamental theorem.

THEOREM 1. The set $N$ of nilpotents of a generalized periodic ring $R$ is an ideal of $R$. 
PROOF. By Lemma 1 (iii), (ii), we have
\[ N \subseteq J \subseteq N \cup C. \] (2.4)
Let \( a \in N, b \in N. \) Then \( a \in J, b \in J, \) \( a - b \in J, \) and hence [see (2.4)] \( a - b \in N \) or \( a - b \in C. \) If \( a - b \in C, \) then \( ab = ba \) and hence \( a - b \in N. \) So, in any case, \( a - b \in N \) for all \( a \in N, b \in N. \) We have already established [see 2.2)] that \( ax \in N \) for all \( a \in N, x \in R, \) and a similar argument yields \( xa \in N. \) Therefore, \( N \) is an ideal.

**THEOREM 2.** Let \( R \) be a generalized periodic ring. Then \( R/N \) is commutative, and hence \( C(R) \subseteq N. \)

**PROOF.** By Theorem 1, \( N \) is an ideal, and hence \( R/N \) makes sense. Let \( x \in R, x \notin C. \) Then, by (1.1),
\[ x^n - x^m \in N, \] for some \( n > m, \) say. (2.5)
It is readily verified that
\[ \left(x^{n-m+1} - x\right)^m = \left(x^{n-m+1} - x\right)x^{m-1}g(x), \text{ some } g(\lambda) \in Z[\lambda], \]
and hence
\[ x^{n-m+1} - x \in N \] (since \( x^n - x^m \in N). \]
Therefore, for all \( x \in R, \) we have
\[ x - x^{n-m+1} \in N \] or \( x \notin C, \) \( n > m, \) (\( x \in R). \) (2.6)
Hence, \( R/N \) has the property that for each \( x \in R/N, \) there exists \( k > 1 \) for which \( x - x^k \) is central. By a well known theorem of Herstein [1], it follows that \( R/N \) is commutative, and thus \( C(R) \subseteq N. \)

Since \( N \) is an ideal of \( R \) (Theorem 1), therefore \( N \subseteq J. \) Combining this with \( C(R) \subseteq N \) and Lemma 1 (ii) we obtain

**LEMMA 2.** Let \( R \) be a generalized periodic ring. Then
\[ C(R) \subseteq N \subseteq J \subseteq N \cup C. \] (2.7)

Next, we consider a ring which is both weakly periodic and generalized periodic.

**THEOREM 3.** If a ring \( R \) is both generalized periodic and weakly periodic, then \( R \) is periodic.

**PROOF.** Let \( x \in R. \) Since \( R \) is weakly periodic, we have
\[ x = a + b \] for some \( a \in N, b \text{ potent } \) (\( b^q = b, q > 1). \) (2.8)
Thus, \( x - a = (x - a)^q; \) and since \( N \) is an ideal, we have \( x - x^q \in N. \) By a well known theorem of Chacron [2], it follows that \( R \) is periodic.

3. **COMMUTATIVITY OF GENERALIZED PERIODIC RINGS.**

We now turn our attention to some conditions which, when imposed on a generalized periodic ring, render it commutative. We begin with the following result, which is suggested by an old theorem on periodic rings.

**THEOREM 4.** Let \( R \) be a generalized periodic ring, and suppose \( N \subseteq C. \) Then \( R \) is commutative.

**PROOF.** By (2.6), for each \( x \in R, \) either \( x \in C \) or \( x - x^k \in N \) for some \( k > 1. \) Since \( N \subseteq C, \) therefore, for every \( x \in R, x - x^k \in C \) for some \( k > 1. \) Therefore, by Herstein's Theorem [1], \( R \) is commutative.

**COLLARY 1.** Let \( R \) be a generalized periodic ring, and suppose \( J \subseteq C. \) Then \( R \) is commutative.

**PROOF.** By Lemma 2, \( N \subseteq J, \) and hence \( N \subseteq C. \) Thus, \( R \) is commutative, by Theorem 4.

**COLLARY 2.** Let \( R \) be a generalized periodic ring with Jacobson radical \( J. \) Then \( J = N \) or \( R \) is commutative.

**PROOF.** By Lemma 1 (ii), it follows that
\[ J = (J \cap N) \cup (J \cap C). \] (3.1)
Viewing (3.1) as a relation holding on additive subgroups, we conclude that
\[ J = J \cap N \text{ or } J = J \cap C. \quad (3.2) \]
This implies that
\[ J \subseteq N \text{ or } J \subseteq C. \quad (3.3) \]
If \( J \subseteq N \), then \( J = N \) [see (2.7)]. On the other hand, if \( J \subseteq C \), then \( R \) is commutative, by Corollary 1.

COROLLARY 3. Let \( R \) be a generalized periodic ring which is not commutative. Then \( J \) coincides with \( N \).

Before stating the next theorem, let us first consider the following two examples, which show that neither centrality of idempotents nor commutativity of nilpotent elements implies commutativity of a generalized periodic ring. Note that, in each case, central elements are zero divisors.

EXAMPLE 1. Let
\[ R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} | 0, 1 \in \text{GF}(2) \right\}. \]
It is readily verified that \( R \) is a generalized periodic ring with commuting nilpotents but its idempotents are not in the center.

EXAMPLE 2. Let
\[ R = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} | a, b, c \in \text{GF}(3) \].
It can be seen that \( R \) is, again a generalized periodic ring with central idempotents but its nilpotents do not commute with each other.

Experience shows that a condition which does not imply commutativity for general rings may do so for rings with \( 1 \). Indeed, we can show that generalized periodic rings with \( 1 \) are commutative; in fact, in the following theorem, we can do better than that.

THEOREM 5. Suppose that \( R \) is a generalized periodic ring containing a central element which is not a zero divisor. Then \( R \) is commutative.

PROOF. In view of Theorem 4, we need only show that \( N \subseteq C \). Suppose not, and choose \( a_0 \in N \setminus C \). Let \( \sigma_0 > 1 \) be the minimal positive integer for which \( a_0^\sigma \in C \) for all \( \sigma \geq \sigma_0 \); and let \( a = a_0^{\sigma_0-1} \). Note that \( a \in C \), and \( a^\lambda \in C \) for all \( \lambda \geq 2 \). Now if \( c \in C \) is not a zero divisor, then \( c + a \in N \cup C \), so there exist \( n, m \) of opposite parity with \( n > m \), such that
\[ (c + a)^n - (c + a)^m \in N \cap C, \quad (n > m). \quad (3.4) \]
We shall assume that \( n \) is even and \( m \) is odd, the other case being only marginally different.

From (3.4) we have \( nc^{n-1}a - mc^{m-1}a \in C \), from which it follows that (since \( c \) is not a zero divisor)
\[ nc^{n-m}a - ma \in C. \quad (3.5) \]
Another consequence of (3.4) is that \( c^n - c^m \in N \) and hence \( c^j - c \in N \), where \( j \) is the even integer \( n - m + 1 \). Replacing \( c \) by \( -c \) in our argument, we also get an even integer \( k \) such that \( (-c)^k - (-c) \in N \). Since \( N \) is an ideal, we have \( c^{1+s(j-1)} - c \in N \) and \( (-c)^{1+((k-1)/t)} - (-c) \in N \) for all positive integers \( s \) and \( t \); and taking \( q = 1 + (j-1)(k-1) \), we see that \( q \) is even, \( c^q - c \in N \) and \( (-c)^q - (-c) \in N \). It follows at once that \( 2c \in N \) and hence \( 2^r c' = 0 \) for some positive integer \( r \). Since \( c \) is not a zero divisor, this yields \( 2^r R = \{0\} \); and, in particular,
\[ 2^r a \in C. \quad (3.6) \]
By hypothesis, \( n \) is even, say \( n = 2n_0 \), and hence (3.5) yields
Therefore, using (3.7) we see that
\[ m^2a = 2n_0c^{n-m}a + mz, \quad z \in C. \] (3.7)
and proceeding inductively, we get (see (3.6))
\[ m^2a \in C. \] (3.8)
Since \( m \) was odd, (3.6) and (3.8) are incompatible with the assumption that \( a \in C \). Therefore \( N \subset C \), as required. This proves the theorem.

**COLLARY 4.** Let \( R \) be a ring with 1. If \( R \) is generalized periodic, then \( R \) is commutative.

**COLLARY 5.** Let \( R \) be a prime ring with nonzero center. If \( R \) is generalized periodic, then \( R \) is commutative.

Our final theorem confronts the impediments of Examples 1 and 2 in a more direct way.

**THEOREM 6.** Suppose \( R \) is a generalized periodic ring, \( N \) the set of nilpotents, and \( E \) the set of idempotents of \( R \). Suppose that
(i) \( E \subset C \) (center of \( R \)); and
(ii) Every commutator \([a, b] = ab - ba\) with \( a \in N \) and \( b \in N \) is potent
(i.e., \([a, b]^q = [a, b]\) for some \( q > 1 \)).

Then \( R \) is commutative.

**PROOF.** By (2.7), \( C(R) \subseteq N \), and hence \([a, b] \in N \). By hypothesis, \([a, b] = [a, b]^q = [a, b]^{1+\lambda(q-1)}\) for all positive integers \( \lambda \), and hence \([a, b] = 0 \) (since \([a, b] \in N \)). Thus,
\[ [a, b] = 0 \text{ for all } a, b \in N \quad \text{i.e., } N \text{ is commutative.} \] (3.9)
Recall also that, in (2.6), we proved that, for ever \( x \) in \( R \), we have
\[ x - x^k \in N \text{ for some } k > 1, \text{ or } x \in C, \quad (x \in R). \] (3.10)
Combining (3.9), (3.10), we see that
\[ \text{For all } x, y \text{ in } R, \left[x - x^k, y - y^r\right] = 0 \text{ for some } k > 1, r > 1. \] (3.11)
As is well known,
\[ R \equiv a \text{ subdirect sum of subdirectly irreducible rings } R_i, \quad (i \in \Gamma). \] (3.12)
We now take a closer look at the structure of each of these subdirect summands \( R_i \), with an eye towards proving their commutativity.

**CASE 1:** \( R_i \) does not have an identity.

Let \( \sigma: R \to R_i \) be the natural homomorphism of \( R \) onto \( R_i \), and let \( \sigma: x \to x_i \). Let \( N_i \) and \( C_i \) denote the set of nilpotents and the center of \( R_i \), respectively. We claim that
\[ R_i \subset N_i \cup C_i. \] (3.13)
Suppose not. Let \( x_i \in R_i \), \( x_i \notin N_i \), \( x_i \notin C_i \), and let \( \sigma: x \to x_i \), \( (x \in R) \). Then, clearly, \( x \notin N \) and \( x \notin C \), and hence by (1.1),
\[ x^n - x^m \in N \text{ for some positive integers } n \text{ and } m, n \neq m. \]
This implies (see the proof of Lemma 1 (ii)) that
\[ x^q = x^q e \text{ for some positive integer } q \text{ and some idempotent } e \text{ in } R. \]
By hypothesis (i), e is a central idempotent, and hence
\[ x^9 = x^9e, \quad e^2 = e \in C. \]
This implies, in \( R_i \), that
\[ x_i^q = x_i^q e_i, \quad e_i^2 = e_i \in C_i. \]  
(3.14)
Since \( e_i \) is a central idempotent in the subdirectly irreducible ring \( R_i \), therefore \( e_i = 0 \) (recall that \( R_i \) does not have an identity), and hence by (3.14), \( x_i^q = 0 \), a contradiction, since \( x_i \) is not nilpotent. This contradiction proves (3.13).

Returning to (3.11), we see that
\[ [x_i - x_i^r, y_i - y_i^r] = 0; \quad k > 1, r > 1; \quad x_i, y_i \in R_i \text{ (arbitrary).} \]  
(3.15)
Now, by a trivial minimality argument, it is readily verified that (3.15) implies:
\[ [a_i, b_i] = 0 \text{ for all nilpotents } a_i, b_i \text{ in } R_i; \text{ (i.e., } N_i \text{ is commutative).} \]  
(3.16)
Combining (3.13) and (3.16), we see that \( R_i \) is commutative.

CASE 2: \( R_i \) has an identity.

Since the homomorphic image of a generalized periodic ring is also generalized periodic, it follows that \( R_i \) is commutative, by Corollary 4.

Since each \( R_i \) in the subdirect sum representation (3.12) is commutative, therefore the ground ring \( R \) itself is also commutative, and the theorem is proved.

COLLARY 6. Any generalized periodic ring with central idempotents and commuting nilpotents is commutative.

REFERENCES