FIXED POINT THEOREMS FOR NON-SELF MAPS IN d-COMPLETE TOPOLOGICAL SPACES

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ABSTRACT. Fixed point theorems are given for non-self maps and pairs of non-self maps defined on d-complete topological spaces.

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1. INTRODUCTION.

Let \((X, t)\) be a topological space and \(d : X \times X \to [0, \infty)\) such that \(d(x, y) = 0\) if and only if \(x = y\). \(X\) is said to be d-complete if \(\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty\) implies that the sequence \(\{x_n\}_{n=1}^{\infty}\) is convergent in \((X, t)\). Complete metric spaces and complete quasi-metric spaces are examples of d-complete topological spaces. The d-complete semi-metric spaces form an important class of examples of d-complete topological spaces.

Let \(X\) be an infinite set and \(t\) any \(T_1\) non-discrete first countable topology for \(X\). There exists a complete metric \(d\) for \(X\) such that \(t \subseteq t_d\) and the metric topology \(t_d\) is non-discrete. Now \((X, t, d)\) is d-complete since \(\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty\) implies that \(\{x_n\}_{n=1}^{\infty}\) is Cauchy in \(t_d\). Thus, \(x_n \to x\), as \(n \to \infty\), in \(t_d\) and therefore in the topology \(t\). The construction of \(t_d\) is given by T. L. Hicks and W. R. Crisler in [1].

Recently, T. L. Hicks in [2] and T. L. Hicks and B. E. Rhoades in [3] and [4] proved several metric space fixed point theorems in d-complete topological spaces. We shall prove additional theorems in this setting.

Let \(T : X \to X\) be a mapping. \(T\) is \(\omega\)-continuous at \(x\) if \(x_n \to x\) implies \(Tx_n \to Tx\) as \(n \to \infty\).

A real-valued function \(G : X \to [0, \infty)\) is lower semi-continuous if and only if \(\{x_n\}_{n=1}^{\infty}\) is a sequence in \(X\) and \(\lim_{n \to \infty} x_n = p\) implies \(G(p) \leq \liminf_{n \to \infty} G(x_n)\).

2. RESULTS.

In [2], Hicks gave the following result.

**THEOREM ([2], Theorem 2):** Suppose \(X\) is a d-complete Hausdorff topological space, \(T : X \to X\) is \(\omega\)-continuous and satisfies \(d(Tx, T^2x) \leq k(d(x, Tx))\) for all \(x \in X\), where 
\(k : [0, \infty) \to [0, \infty)\), \(k(0) = 0\), and \(k\) is non-decreasing. Then \(T\) has a fixed point if and only if
there exists \( x \) in \( X \) with \( \sum_{n=1}^{\infty} k^n(d(x, Tx)) < \infty \). In this case, \( x_n = T^n x \to p = Tp \). [\( k \) is not assumed to be continuous and \( k^2(a) = k(k(a)) \).]

The following conditions are examined. Let \( T : C \to X \) with \( C \) a closed subset of the d-complete topological space \( X \) and \( C \subset T(C) \). Let \( k : [0, \infty) \to [0, \infty) \) be such that \( k(0) = 0, k \) is non-decreasing, and

\[
k(d(Tx, Ty)) \geq d(x, y) \tag{2.1}
\]

for all \( x, y \in C \), or

\[
d(Tx, Ty) \geq k(d(x, y)) \tag{2.2}
\]

for all \( x, y \in C \), or

\[
d(x, y) \geq k(d(Tx, Ty)) \tag{2.3}
\]

for all \( x, y \in C \), or

\[
k(d(x, y)) \geq d(Tx, Ty) \tag{2.4}
\]

for all \( x, y \in C \).

It will be shown that condition (2.1) leads to a fixed point, but that the other three conditions do not guarantee a fixed point.

**THEOREM 1.** Suppose \( X \) is a d-complete Hausdorff topological space, \( C \) is a closed subset of \( X \), and \( T : C \to X \) is an open mapping with \( C \subset T(C) \) which satisfies \( d(x, y) \leq k(d(Tx, Ty)) \) for all \( x, y \in C \) where \( k : [0, \infty) \to [0, \infty) \), \( k(0) = 0 \), and \( k \) is non-decreasing. Then \( T \) has a fixed point if and only if there exists \( x_0 \in C \) with \( \sum_{n=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty \).

**PROOF.** Notice that the condition \( d(x, y) \leq k(d(Tx, Ty)) \) forces \( T \) to be one-to-one. Hence \( T^{-1} \) exists. Also, \( T \) is open implies that \( T^{-1} \) is continuous, and thus \( \omega \)-continuous.

If \( p = Tp \) then \( \sum_{n=1}^{\infty} k^n(d(Tp, p)) = 0 < \infty \).

Suppose there exists \( x_0 \in C \) such that \( \sum_{n=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty \). We know that \( T^{-1} \) exists, so let \( T_1 \) be \( T^{-1} \) restricted to \( C \). Then \( T_1 : C \to C \) and \( d(T_1 x, T_1 y) \leq k(d(x, y)) \) for all \( x, y \in C \). Let \( y = T_1 x \). Then \( d(T_1 x, T_1^2 x) \leq k(d(x, T_1 x)) \) for all \( x \in C \). In particular,

\[
d(T_1^2 x_0, T_1 x_0) \leq k(d(x_0, T_1 x_0)) \leq k^2(d(Tx_0, x_0)) \tag{2.3}
\]

by induction.

\[
d(T_1^n x_0, T_1^n x_0) \leq k^n(d(Tx_0, x_0)).
\]

Thus,

\[
\sum_{k=1}^{\infty} d(T_1^{n-1} x_0, T_1^n x_0) \leq \sum_{k=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty.
\]

Since \( X \) is d-complete, \( T_1^n x_0 \) converges, say to \( p \). Note that \( p \) is in \( C \) since \( C \) is closed. Now \( T_1(T_1^n x_0) \to T_1 p \) as \( n \to \infty \) since \( T_1 \) is \( \omega \)-continuous. But \( T_1^{n+1} x_0 \to p \) as \( n \to \infty \), and since limits are unique in \( X \), \( T_1 p = p \). Now \( T(T_1 p) = T(p) \) and \( T(T_1 p) = p \) so \( Tp = p \) and \( T \) has a fixed point.

**COROLLARY 1.** Suppose \( T : C \to X \) where \( C \) is a closed subset of a d-complete Hausdorff symmetric topological space with \( C \subset T(C) \). Suppose \( d(x, y) \leq [d(Tx, Ty)]^p \) where \( p > 1 \) for all \( x, y \in C \). If there exists \( x_0 \in C \) such that \( d(Tx_0, x_0) < 1 \), then \( T \) has a fixed point.

**PROOF.** If \( x \neq y, 0 < d(x, y) \leq [d(Tx, Ty)]^p \) and \( Tx \neq Ty \). Thus \( T \) is one-to-one and \( T^{-1} \) exists. Now \( d(T^{-1} x, T^{-1} y) \leq [d(x, y)]^p \) implies that \( T^{-1} \) is continuous. Hence \( T \) must be
open. Let \( x_0 \) be a point in \( C \) such that \( d(Tx_0, x_0) < 1 \). If \( d(Tx_0, x_0) = 0 \), then \( x_0 \) is a fixed point of \( T \). Suppose \( 0 < d(Tx_0, x_0) < 1 \). Let \( k(t) = t^p \), and \( t = d(Tx_0, x_0) \). Note that \((\alpha t)^p < \alpha t^p \) if \( 0 < \alpha < 1 \). Since \( t^p < t \) there is an \( \alpha_1 \in (0, 1) \) such that \( t^p = \alpha_1 t \). Now \((t^p)^p < t^p \) and there is an \( \alpha_2 \in (0, 1) \) such that \( \alpha_2 t = \alpha_2 t^p = t^p \). But \( \alpha_2 t^p = t^2 = (t^p)^p = (\alpha_1 t)^p = \alpha_1 t t^p \). Hence \( \alpha_2 < \alpha_1 \).

Now \( t^p = \alpha_2 t^p = \alpha_2 t \). Assume \( t^{mp} < \alpha t^{mp} \). Then \( t^{np} = (t^{mp})^p < (\alpha t^{mp})^p = \alpha t^{np} \). Hence, by induction, \( t^{np} < \alpha t^{np} \) for all natural numbers \( n \).

Therefore,

\[
\sum_{n=1}^{\infty} k^p(d(Tx_0, x_0)) = \sum_{n=1}^{\infty} [d(Tx_0, x_0)]^{mp} = \sum_{n=1}^{\infty} t^{np} < \sum_{n=1}^{\infty} \alpha^p < \infty
\]

since \( 0 < \alpha_1 < 1 \). Applying Theorem 1, we get that \( T \) has a fixed point.

If \( T \) is not open one could check the following condition.

**THEOREM 2.** Let \( X \) be a \( d \)-complete Hausdorff topological space, \( C \) be a closed subset of \( X \), \( T : C \to X \) with \( C \subseteq T(C) \). Suppose there exists \( k : [0, \infty) \to [0, \infty) \) such that \( k(d(Tx, Ty)) \geq d(x, y) \) for all \( x, y \in C \), \( k \) is non-decreasing, \( k(0) = 0 \), and there exists \( x_0 \in C \) such that \( \sum_{n=1}^{\infty} k^p(d(Tx_0, x_0)) < \infty \). If \( G(x) = d(Tx, x) \) is lower semi-continuous on \( C \) then \( T \) has a fixed point.

**PROOF.** If \( x \neq y \), \( 0 < d(x, y) \leq d(Tx, Ty) \) so that \( d(Tx, Ty) \neq 0 \). Hence \( T \) is one-to-one and \( T^{-1} \) exists. Let \( T_1 \) be \( T^{-1} \) restricted to \( C \). Now \( T_1 : C \to C \) and for \( x \in C \),

\[
d(x, T_1 x) \leq k(d(Tx, x)) \quad \text{and} \quad d(T_1 x, T_2 x) \leq k(d(x, T_1 x)) \leq d(Tx, x).
\]

By induction,

\[
d(T_1^{n-1} x, T_1^n x) \leq k^p(d(Tx, x)).
\]

There exists \( x_0 \in C \) with \( \sum_{n=1}^{\infty} k^p(d(Tx_0, x_0)) < \infty \) implies \( \sum_{n=1}^{\infty} d(T_1^{n-1} x_0, T_1^n x_0) < \infty \). Since \( X \) is \( d \)-complete there exists \( p \in X \) such that \( T_1^n x_0 \to p \) as \( n \to \infty \). Note that \( p \in C \) since \( T_1^n x_0 \in C \) for all \( n \) and \( C \) is closed. Now \( G(x) = d(Tx, x) \) is lower semi-continuous on \( C \) gives \( G(p) \leq \lim \inf G(T_1^n x_0) \) or \( d(Tp, p) \leq \lim \inf d(T_1^{n-1} x_0, T_1^n x_0) = 0 \). Thus \( Tp = p \).

In [5], Hicks gives several examples of functions \( k \) which satisfy the condition of theorem 1 of that paper. These examples, with a slight modification, carry over to the non-self map case. The non-self map version of Example 1 is given for completeness. The other examples carry over in a similar manner.

**EXAMPLE 1.** Suppose \( 0 < \lambda < 1 \). Let \( k(t) = \lambda t \) for \( t \geq 0 \). If \( d(x, y) \leq \lambda d(Tx, Ty) \), \( T \) is open since \( T^{-1} \) exists and is continuous. Let \( x \in C \). There exists \( y \in C \) such that \( Ty = x \). Now \( d(x, y) = d(Ty, y) \leq \lambda d(T^2y, Ty) \) and \( \sum_{n=1}^{\infty} k^p(d(Ty, y)) \leq \sum_{n=1}^{\infty} \lambda^p d(T^2y, Ty) < \infty \). Applying Theorem 1 we get a fixed point for \( T \). (Note: \( d(x, y) \leq \lambda d(Tx, Ty) \) for \( 0 < \lambda < 1 \) is equivalent to \( d(Tx, Ty) \geq \alpha d(x, y) \) for \( \alpha > 1 \)).

The following examples show that conditions (2.2), (2.3) and (2.4) do not guarantee fixed points.

**EXAMPLE 2.** Let \( \mathbb{R} \) denote the real numbers and \( CB(\mathbb{R}, \mathbb{R}) \) denote the collection of all bounded and continuous functions which map \( \mathbb{R} \) into \( \mathbb{R} \). Let

\[
C = \{ f \in CB(\mathbb{R}, \mathbb{R}) : f(t) = 0 \text{ for all } t \leq 0 \text{ and } \lim_{t \to \infty} f(t) \geq 1 \}.
\]
Define $T : C \to CB(\mathbb{R}, \mathbb{R})$ by $Tf(t) = \frac{1}{2}f(t + 1)$ and let $k(t) = \frac{t}{3}$. Then $d(Tf, Tg) = \frac{1}{2}d(f, g) \geq k(d(f, g))$. $k$ satisfies condition (2.2) but, as shown in [6], $T$ does not have a fixed point.

**EXAMPLE 3.** Let $T : [1, \infty) \to [0, \infty)$ be defined by $Tx = x - \frac{1}{x}$ and let $k(t) = \frac{1}{2}$. Then $d(Tx, Ty) \leq 2d(x, y)$ or $d(x, y) \geq k(d(Tx, Ty))$. $k$ satisfies condition (2.3) but $T$ does not have a fixed point.

**EXAMPLE 4.** Let $c_0$ denote the collection of all sequences that converge to zero. Let $C = \{x \in c_0 : \|x\|_1 = 1 \text{ and } x_0 = 1\}$. Define $T : C \to c_0$ by $Tx = y$ where $y_n = x_{n+1}, \quad n = 0, 1, 2, \ldots,$ and let $k(t) = 2t$. Then $d(Tx, Ty) = d(x, y) \geq 2d(x, y) = k(d(x, y))$ for all $x, y \in C$. $k$ satisfies condition (2.4) but, as shown in [6], $T$ does not have a fixed point.

The following theorems were motivated by the work of Hicks and Rhoades [3].

**THEOREM 3.** Let $C$ be a compact subset of a Hausdorff topological space $(X, t)$ and $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. Suppose $T : C \to X$ with $C \subset T(C)$, $T$ and $G(x) = d(x, Tx)$ are both continuous, and $d(Tx, T^2x) > d(x, Tx)$ for all $x \in T^{-1}(C)$ with $x \neq Tx$. Then $T$ has a fixed point in $C$.

**PROOF.** $C$ is a compact subset of a Hausdorff space so it is closed. $T$ is continuous so $T^{-1}(C)$ is closed and hence is compact since $T^{-1}(C) \subset C$. $G(x)$ is continuous so it attains its minimum on $T^{-1}(C)$, say at $z$. Now $z \in C \subset T(C)$ so there exists $y \in T^{-1}(C)$ such that $Ty = z$. If $y \neq z$ then $d(z, Ty) = d(Ty, T^2y) > d(y, Ty)$, a contradiction. Thus $y = z = Ty$ is a fixed point of $T$.

**THEOREM 4.** Let $C$ be a compact subset of a Hausdorff topological space $(X, t)$ and $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. Suppose $T : C \to X$ with $C \subset T(C)$, $T$ and $G(x) = d(x, Tx)$ are both continuous, $f : [0, \infty) \to [0, \infty)$ is continuous and $f(t) > 0$ for $t \neq 0$. If we know that $d(Tx, T^2x) \leq \lambda f(d(x, Tx))$ for all $x \in T^{-1}(C)$ implies $T$ has a fixed point where $0 < \lambda < 1$, then $d(Tx, T^2x) < f(d(x, Tx))$ for all $x \in T^{-1}(C)$ such that $f(d(x, Tx)) \neq 0$ gives a fixed point.

**PROOF.** $C$ is a compact subset of a Hausdorff space so it is closed. $T$ is continuous gives that $T^{-1}(C)$ is closed, and $T^{-1}(C) \subset C$ so $T^{-1}(C)$ is compact. Suppose $x \neq Tx$ for all $x \in T^{-1}(C)$. Then $d(x, Tx) > 0$ so that $f(d(x, Tx)) > 0$ for all $x \in T^{-1}(C)$. Define $P(x)$ on $T^{-1}(C)$ by $P(x) = \frac{d(Tx, T^2x)}{f(d(x, Tx))}$. $P$ is continuous since $T$, $f$ and $G(x)$ are continuous. Therefore $P$ attains its maximum on $T^{-1}(C)$, say at $z$. $P(x) \leq P(z) < 1$ so $d(Tx, T^2x) \leq P(z)f(d(x, Tx))$ and $T$ must have a fixed point.

**THEOREM 5.** Let $C$ be a compact subset of a Hausdorff topological space $(X, t)$ and $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. Suppose $T : C \to X$ with $C \subset T(C)$, $T$ and $G(x) = d(x, Tx)$ are both continuous, $f : [0, \infty) \to [0, \infty)$ is continuous and $f(t) > 0$ for $t \neq 0$. If we know that $d(Tx, T^2x) \geq \lambda f(d(x, Tx))$ for all $x \in T^{-1}(C)$ implies $T$ has a fixed point where $\lambda > 1$, then $d(Tx, T^2x) > f(d(x, Tx))$ for all $x \in T^{-1}(C)$ such that $f(d(x, Tx)) \neq 0$ gives a fixed point.
PROOF. C is a compact subset of a Hausdorff space so it is closed. T is continuous gives that \( T^{-1}(C) \) is closed and hence compact, since \( T^{-1}(C) \subset C \). Suppose \( x \neq Tx \) for all \( x \in T^{-1}(C) \). Then \( d(x, Tx) > 0 \) and \( f(d(x, Tx)) > 0 \). Define \( P(x) = \frac{d(Tx, T^2x)}{f(d(x, Tx))} \). \( P \) is continuous since \( T \), \( f \) and \( G \) are continuous. \( P \) attains its minimum on \( T^{-1}(C) \), say at \( x \). \( P(x) \geq P(z) > 1 \) so \( d(Tx, T^2x) \geq P(z)f(d(x, Tx)) \) and \( T \) must have a fixed point.

Theorems 6, 7 and 8 are generalizations of theorems by Kang [7]. The following family of real functions was originally introduced by M. A. Khan, M. S. Khan, and S. Sessa in [8]. Let \( \Phi \) denote the family of all real functions \( R^3 \rightarrow R^+ \) satisfying the following conditions:

- \((C_1)\) \( \phi \) is lower-semicontinuous in each coordinate variable,
- \((C_2)\) Let \( v, w \in R^+ \) be such that either \( v \geq \phi(v, w, w) \) or \( v \geq \phi(w, v, w) \). Then \( v \geq hw \), where \( \phi(1, 1, 1) = h > 1 \).

THEOREM 6. Let \((X, t, d)\) be a \( d \)-complete topological space where \( d \) is a continuous symmetric. Let \( A \) and \( B \) map \( C \), a closed subset of \( X \), into (onto) \( X \) such that \( C \subset A(C) \), \( C \subset B(C) \), and \( d(Ax, By) \geq \phi(d(Ax, x), d(By, y), d(x, y)) \) for all \( x, y \in C \) where \( \phi \in \Phi \). Then \( A \) and \( B \) have a common fixed point in \( C \).

PROOF. Fix \( x_0 \in C \). Since \( C \subset A(C) \) there exists \( x_1 \in C \) such that \( Ax_1 = x_0 \). Now \( C \subset B(C) \) so there exists \( x_2 \in C \) such that \( Bx_2 = x_1 \). Build the sequence \( \{x_n\}^\infty_{n=0} \) by \( Ax_{2n+1} = x_{2n}, \ Bx_{2n+2} = x_{2n+1} \). Now if \( x_{2n+1} = x_{2n} \) for some \( n \), the \( x_{2n+1} \) is a fixed point of \( A \). Then

\[
\begin{align*}
d(x_{2n+1}, x_{2n+1}) &= d(x_{2n}, x_{2n+1}) \\
&= d(Ax_{2n+1}, Bx_{2n+2}) \\
&\geq \phi(d(Ax_{2n+1}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})) \\
&= \phi(0, d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}))
\end{align*}
\]

By property \((C_2)\), \( d(x_{2n}, x_{2n+1}) \geq h d(x_{2n+1}, x_{2n+2}) \). Hence, \( x_{2n+1} = x_{2n+2} \) and \( Bx_{2n+1} = Bx_{2n+2} = x_{2n+1} \). Therefore \( x_{2n+1} \) is a common fixed point of \( A \) and \( B \). Now if \( x_{2n+1} = x_{2n+2} \) for some \( n \), then \( Bx_{2n+2} = Bx_{2n+1} = x_{2n+2} \). Then

\[
\begin{align*}
d(x_{2n+2}, x_{2n+1}) &= d(Ax_{2n+3}, Bx_{2n+2}) \\
&\geq \phi(d(Ax_{2n+3}, x_{2n+3}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+3}, x_{2n+2})) \\
&= \phi(d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+3}, x_{2n+2})).
\end{align*}
\]

By property \((C_2)\), \( d(x_{2n+1}, x_{2n+2}) \geq h d(x_{2n+2}, x_{2n+3}) \) or \( x_{2n+2} = x_{2n+3} \). Thus \( Ax_{2n+2} = Ax_{2n+3} = x_{2n+2} \) and \( x_{2n+2} \) is a fixed point of \( A \) also.

Suppose \( x_n \neq x_{n+1} \) for all \( n \). Then

\[
\begin{align*}
d(x_{2n}, x_{2n+1}) &= d(Ax_{2n+1}, Bx_{2n+2}) \\
&\geq \phi(d(Ax_{2n+1}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})) \\
&= \phi(d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})).
\end{align*}
\]
Again by (C2), \(d(x_{2n}, x_{2n+1}) \geq \frac{1}{h} d(x_{2n+1}, x_{2n+2})\) or \(d(x_{2n} + 1, x_{2n+2}) \leq \frac{1}{h} d(x_{2n}, x_{2n+1})\).

Also,
\[
d(x_{2n+1}, x_{2n+2}) = d(Ax_{2n+1}, Bx_{2n+2}) \\
\geq \phi(d(Ax_{2n+1}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})) \\
= \phi(d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}), d(x_{2n+2}, x_{2n+2})).
\]

By (C2) we get \(d(x_{2n+2}, x_{2n+3}) \leq \frac{1}{h} d(x_{2n+1}, x_{2n+2})\). Induction gives
\[
d(x_{n+1}, x_{n+2}) \leq \frac{1}{h^n} d(x_0, x_1).
\]
Thus \(\sum_{n=1}^{\infty} d(x_{n+1}, x_{n+2}) \leq \sum_{n=1}^{\infty} \left(\frac{1}{h}\right)^{n+1} d(x_0, x_1) < \infty\). X is \(h\)-complete so \(x_n \to p\) as \(n \to \infty\) where \(p \in C\), since \(C\) is closed. We also have \(x_{2n} \to p\) and \(x_{2n+1} \to p\) as \(n \to \infty\). This gives \(Ax_{2n+1} \to p\) and \(Bx_{2n+2} \to p\) as \(n \to \infty\). Since \(p \in C\), \(p \in A(C)\) and \(p \in B(C)\), so there exist \(v, w \in C\) such that \(Av = p\) and \(Bw = p\). Now
\[
d(x_{2n}, p) = d(Ax_{2n+1}, Bw) \\
\geq \phi(d(Ax_{2n+1}, x_{2n+1}), d(Bw, w), d(x_{2n+1}, w)).
\]
Since \(\phi\) is lower-semicontinuous, letting \(n \to \infty\) gives \(d(p, p) \geq \phi(0, d(p, w), d(p, w))\) and by (C2) we have \(0 \geq h d(p, w)\). Hence \(p = w\). Also,
\[
d(p, x_{2n+1}) = d(Av, Bx_{2n+1}) \geq \phi(d(Av, v), d(Bx_{2n+2}, x_{2n+2}), d(v, x_{2n+1})).
\]
Letting \(n \to \infty\) gives \(d(p, p) \geq \phi(d(p, v), 0, d(v, p))\) or, by (C2), \(0 \geq h d(p, v)\). Hence \(p = v\). Therefore, \(Ap = Av = p = Bw = Bp\).

COROLLARY 2. Let A and B map C, a closed subset of X, into (onto) X such that
\(C \subset A(C)\), \(C \subset B(C)\), and \(d(Ax, By) \geq a d(Ax, x) + b d(By, y) + c d(x, y)\) for all \(x, y \in C\), where \(a, b, c\) are non-negative real numbers with \(a < 1, b < 1,\) and \(a + b + c > 1\). Then A and B have a common fixed point in C.

The proof of Corollary 2 is identical to the proof of Corollary 2.3 in [9].

In [7], Kang defined \(\Phi^*\) to be the family of all real functions \(\varphi \to (R^+)^3 \to R^+\) satisfying condition (C1) and the following condition:
(C3) Let \(v, w \in R^+ - \{0\}\) be such that either \(v \geq \varphi(v, w, w)\) or \(v \geq \varphi(w, v, v)\). Then \(v \geq hw\), where \(\varphi(1, 1, 1) = h > 1\). Kang showed that the family \(\Phi^*\) is strictly larger than the family \(\Phi\).

THEOREM 7. Let \((X, t, d)\) be a d-complete Hausdorff topological space where d is a continuous symmetric. If A and B are continuous mappings from C, a closed subset of X, into X such that \(C \subset A(C)\), \(C \subset B(C)\), and \(d(Ax, By) \geq \varphi(d(Ax, x), d(By, y), d(x, y))\) for all \(x, y \in C\) such that \(x \neq y\) where \(\varphi \in \Phi^*\), then A or B has a fixed point or A and B have a common fixed point.

PROOF. Let \(\{x_n\}_{n=0}^\infty\) be defined as in the proof of Theorem 6. If \(x_n = x_{n+1}\) for some \(n\) then A or B has a fixed point. Suppose \(x_n \neq x_{n+1}\) for all \(n\). As in the proof of Theorem 6, \(x_n \to p\) as \(n \to \infty\). Now \(\{x_{2n}\}_{n=0}^\infty\) and \(\{x_{2n+1}\}_{n=0}^\infty\) are subsequences of \(\{x_n\}_{n=1}^\infty\) and hence each converges to \(p\). Since A and B are continuous, \(Ax_{2n+1} = x_{2n} \to Ap\) and \(Bx_{2n+2} = x_{2n+1} \to Bp\). Limits in X are unique, because X is Hausdorff, so \(Ap = p = Bp\).
COROLLARY 3. Let A and B be continuous mappings from C, a closed subset of X, into X satisfying $C \subseteq A(C)$, $C \subseteq B(C)$ and $d(Ax, By) \geq h \min\{d(Ax, x), d(By, y), d(x, y)\}$ for all $x, y \in C$ with $x \neq y$ where $h > 1$. Then A or B has a fixed point or A and B have a common fixed point.

PROOF. Note that $\varphi(t_1, t_2, t_3) = h \min\{t_1, t_2, t_3\}$, $h > 1$ is in $\Psi^*$. Apply Theorem 7.

If $A = B$ in Corollary 3 we get a generalization of Theorem 3 in [9].

Boyd and Wong [10] call the collection of all real functions $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following conditions $\Psi$:

(C4) $\psi$ is upper-semicontinuous and non-decreasing,

(C5) $\psi(t) < t$ for each $t > 0$.

THEOREM 8. Let $(X, t, d)$ be a $d$-complete symmetric Hausdorff topological space. If A and B are continuous mappings from C, a closed subset of X, into X such that $C \subseteq A(C)$, $C \subseteq B(C)$, and $\psi(d(Ax, By)) \geq \min\{d(Ax, x), d(By, y), d(x, y)\}$ for all $x, y \in C$ where $\psi \in \Psi$ and $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, then either A or B has a fixed point or A and B have a common fixed point.

PROOF. Let $\{x_n\}_{n=0}^{\infty}$ be defined as in the proof of Theorem 6. If $x_n = x_{n+1}$ for some $n$ then A or B has a fixed point. Suppose $x_n \neq x_{n+1}$ for all $n$. Then

$$
\psi(d(x_{2n}, x_{2n+1})) = \psi(d(Ax_{2n+1}, Bx_{2n+2})) \\
\geq \min\{d(Ax_{2n+1}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})\} \\
= \min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})\} \\
= d(x_{2n+1}, x_{2n+2})
$$

since $\psi(t) < t$ for all $t > 0$.

Similarly, $d(x_{2n+2}, x_{2n+3}) \leq \psi(d(x_{2n+1}, x_{2n+2}))$ and hence $d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1}))$ for each $n$. Since $\psi$ is non-decreasing, $d(x_{n+1}, x_{n+2}) \leq \psi^n(d(x_0, x_1))$. Now

$$
\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=0}^{\infty} \psi^n(d(x_0, x_1)) < \infty.
$$

The space X is d-complete so there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$. The mappings A and B are continuous so $Ax_{2n+1} = x_{2n} \to Ap$ and $Bx_{2n+2} = x_{2n+1} \to Bp$. Limits are unique so $Ap = p = Bp$.

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