APPROXIMATION BY WEIGHTED MEANS OF WALSH-FOURIER SERIES

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ABSTRACT. We study the rate of approximation to functions in $L^p$ and, in particular, in $\text{Lip}(\alpha, p)$ by weighted means of their Walsh-Fourier series, where $\alpha > 0$ and $1 < p \leq \infty$. For the case $p = \infty$, $L^p$ is interpreted to be $C_\omega$, the collection of uniformly $W$ continuous functions over the unit interval $[0,1)$. We also note that the weighted mean kernel is quasi-positive, under fairly general conditions.


KEY WORDS AND PHRASES: Walsh system, Walsh-Fourier series, weighted mean, rate of convergence, Lipschitz class, Walsh-Dirichlet kernel, Walsh-Fejér kernel, quasi-positive kernel.

1. INTRODUCTION.

We consider the Walsh orthonormal system $\{w_k(x) : k \geq 0\}$ defined on the unit interval $I := [0,1)$ using the Paley enumeration (see [4]).

Let $P_n$ denote the collection of Walsh polynomials of order less than $n$; that is, functions of the form

$$P(x) := \sum_{k=0}^{n-1} a_k w_k(x),$$

where $n \geq 1$ and $\{a_k\}$ is any sequence of real numbers.

The approximation by Walsh polynomials in the norms of $L^p := L^p(I), 1 \leq p < \infty$, and $C_\omega := C_\omega(I)$. The class $C_\omega$ is the collection of all functions $f : I \rightarrow \mathbb{R}$ that are uniformly continuous from the dyadic topology of $I$ into the usual topology of $\mathbb{R}$; briefly, uniformly $W$-continuous. The dyadic topology is generated by the collection of dyadic intervals of the form

$$I_m := [k2^{-m}, (k + 1)2^{-m}), \quad k = 0, 1, \ldots, 2^m - 1; \quad m = 0, 1 \ldots$$

For $C_\omega$ we shall write $L^{\infty}$. Set

$$\|f\|_p := \left\{ \int_0^1 |f(x)|^p dx \right\}^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty := \sup\{ |f(x)| : x \in I \}.$$

The best approximation of a function $f \in L^p, 1 \leq p \leq \infty$, by polynomials in $P_n$ is defined by

$$E_n(f, L^p) := \inf_{P \in P_n} \|f - P\|_p.$$
For $f \in L^p$, the modulus of continuity is defined by
\[
\omega_p(f, \delta) := \sup \{ \| f(-\delta + t) - f(\cdot) \|_p : |t| < \delta \},
\]
where $\delta > 0$ and $+\delta$ denotes dyadic addition. For $\alpha > 0$, the Lipschitz classes in $L^p$ are defined by
\[
\text{Lip}(\alpha, p) := \{ f \in L^p : \omega_p(f, \delta) = O(\delta^\alpha) \quad \text{as} \quad \delta \to 0 \}.
\]
Concerning further properties and explanations, we refer the reader to [3], whose notations are adopted here as well.

2. MAIN RESULTS.

For $f \in L^1$, its Walsh–Fourier series is defined by
\[
\sum_{k=0}^{\infty} a_k w_k(x), \quad \text{where} \quad a_k := \int_0^1 f(t) w_k(t) dt. \tag{2.1}
\]
The $n$th partial sum of the series in (2.1) is
\[
s_n(f, x) := \sum_{k=0}^{n-1} a_k w_k(x), \quad n \geq 1,
\]
which can also be written in the form
\[
s_n(f, x) = \int_0^1 f(x+t) D_n(t) dt,
\]
where
\[
D_n(t) := \sum_{k=0}^{n-1} w_k(t), \quad n \geq 1,
\]
is the Walsh-Dirichlet kernel of order $n$.

Throughout, $\{p_k : k \geq 1\}$ will denote a sequence of non-negative numbers, with $p_1 > 0$. The weighted means for series (2.1) are defined by
\[
t_n(f, x) := \frac{1}{P_n} \sum_{k=1}^{n} p_k s_k(f, x),
\]
where
\[
P_n := \sum_{k=1}^{n} p_k, \quad n \geq 1.
\]
We shall always assume that
\[
\lim_{n \to \infty} P_n = \infty,
\]
which is the condition for regularity.

The representation
\[
t_n(f, x) = \int_0^1 f(x+t) L_n(t) dt \tag{2.2}
\]
plays a central role in the sequel, where
\[
L_n(t) := \frac{1}{P_n} \sum_{k=1}^{n} p_k D_k(t), \quad n \geq 1, \tag{2.3}
\]
is the weighted mean kernel.
THEOREM 1. Let $f \in L^p, 1 \leq p \leq \infty, n := 2^m + k, 1 \leq k \leq 2^m, m \geq 1.$

(i) If $\{p_k\}$ is nondecreasing and satisfies the condition

$$\frac{n p_n}{P_n} = O(1),$$

then

$$\|t_n(f) - f\|_p \leq \frac{3}{P_n} \sum_{j=0}^{m-1} 2^j p_{2^{j+1}-1} \omega_p(f, 2^{-j}) + O(\omega_p(f, 2^{-m})).$$

(ii) If $\{p_k\}$ is nonincreasing, then

$$\|t_n(f) - f\|_p \leq \frac{3}{P_n} \sum_{j=0}^{m-1} 2^j p_{2^j} \omega_p(f, 2^{-j}) + O(\omega_p(f, 2^{-m})).$$

THEOREM 2. Let $f \in \text{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p \leq \infty.$ Then for $\{p_k\}$ nondecreasing,

$$\|t_n(f) - f\|_p = \begin{cases} O(n^{-\alpha}) & \text{if } 0 < \alpha < 1, \\ O(n^{-1} \log n) & \text{if } \alpha = 1, \\ O(n^{-1}) & \text{if } \alpha > 1; \end{cases}$$

for $\{p_k\}$ nonincreasing,

$$\|t_n(f) - f\|_p = O\left( \frac{1}{P_n} \sum_{j=0}^{m-1} 2^{(1-\alpha)j} p_{2^j} + 2^{-\alpha m} \right).$$

Given two sequences $\{p_k\}$ and $\{q_k\}$ of nonnegative numbers, we write $p_k \asymp q_k$ if there exist two positive constants $C_1$ and $C_2$ such that

$$C_1 q_k \leq p_k \leq C_2 q_k \quad \text{for all } k \text{ large enough.}$$

We present two particular cases for nonincreasing $\{p_k\}$.

Case (i): $p_k \asymp (\log k)^{-\beta}$ for some $\beta > 0.$ Then $P_n \asymp n(\log n)^{-\beta}$. It follows from (2.8)

$$\|t_n(f) - f\|_p = \begin{cases} O(n^{-\alpha}) & \text{if } 0 < \alpha < 1 \text{ and } \beta > 0, \\ O(n^{-1} \log n) & \text{if } \alpha = 1 \text{ and } 0 < \beta < 1, \\ O(n^{-1} \log n \log \log n) & \text{if } \alpha = \beta = 1, \\ O(n^{-1}(\log n)^{\beta}) & \text{if } \alpha = 1 \text{ and } \beta > 1, \text{ or if } \alpha > 1 \text{ and } \beta > 0. \end{cases}$$

Case (ii): $p_k \asymp k^{-\beta}$ for some $0 < \beta \leq 1.$ Then $P_n \asymp n^{1-\beta}$ if $0 < \beta < 1$ and $P_n \asymp \log n$ if $\beta = 1.$ The case $\beta > 1$ is unimportant since $P_n = O(1).$ By (2.8),

$$\|t_n(f) - f\|_p = \begin{cases} O(n^{-\alpha}) & \text{if } \alpha + \beta < 1, \\ O(n^{\beta-1} \log n + n^{-\alpha}) & \text{if } \alpha + \beta = 1, \\ O(n^{\beta-1}) & \text{if } \alpha + \beta > 1 \text{ and } \beta > 1, \\ O((\log n)^{-1}) & \text{if } \beta = 1, \end{cases}$$

where $\alpha > 0$ and $\beta > 0.$

REMARK 1. The slower $P_n$ tends to infinity, the worse is the rate of approximation.

REMARK 2. Watari [6] has shown that a function $f \in L^p$ belongs to $\text{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p \leq \infty$ if and only if

$$E_n(f, L^p) = O(n^{-\alpha}).$$
Thus, for $0 < \alpha < 1$, the rate of approximation to functions $f$ in the class Lip$(\alpha, p)$ by $t_n(f)$ is as good as the best approximation.

REMARK 3. For $\alpha > 1$, the rate of approximation by $t_n(f)$ in the class Lip$(\alpha, p)$ cannot be improved too much.

THEOREM 3. If for some $f \in L^p$, $1 \leq p \leq \infty$, 

$$\|t_{2^m}(f) - f\|_p = o(P_{2^m}^{-1}) \quad \text{as} \quad m \to \infty,$$

then $f$ is a constant.

If $p_k = 1$ for all $k$, then the $t_n(f, x)$ are the $(C, 1)$ - means (i.e., the first arithmetic means) of the series in (2.1). In this case, Theorem 2 was proved by Yano [8] for $0 < \alpha < 1$ and by Jastrebova [1] for $\alpha = 1$; Theorem 3 also reduces to a known result (see e.g. [5, p. 191]).

3. AUXILIARY RESULTS

Let

$$K_n(t) := \frac{1}{n} \sum_{k=1}^{n} D_k(t), \quad n \geq 1$$

by the Walsh-Fejer kernel.

LEMMA 1. (see [7]). Let $m \geq 0$ and $n \geq 1$. Then $K_{2^m}(t) \geq 0$ for each $t \in I$, 

$$\int_0^1 |K_n(t)| dt \leq 2, \quad \text{and} \quad \int_0^1 K_{2^m}(t) dt = 1.$$

LEMMA 2. Let $n := 2^m + k$, $1 \leq k \leq 2^m$, and $m \geq 1$. Then for $L_n(t)$ defined in (2.3),

$$P_n L_n(t) = - \sum_{j=0}^{m-1} r_j(t) w_{2^j-1}(t) \sum_{i=1}^{2^j-1} i(p_{2^{j+1}-i} - p_{2^{j+1}-i-1}) K_i(t)$$

$$- \sum_{j=0}^{m-1} r_j(t) w_{2^j-1}(t) 2^j p_{2^j} K_{2^j}(t)$$

$$+ \sum_{j=0}^{m-1} (p_{2^{j+1}-1} - p_{2^j-1}) D_{2^{j+1}}(t)$$

$$+ (P_n - P_{n-k-1}) D_{2^m}(t) + r_m(t) \sum_{i=1}^{k} p_{2^m+i} D_i(t),$$

where the $r_j(t)$ are the Rademacher functions.

Proof. From (2.3)

$$P_n L_n(t) = \sum_{i=1}^{2^m-1} p_i D_i(t) + \sum_{i=2^m}^{2^m+k} p_i D_i(t)$$

$$= \sum_{j=0}^{m-1} \sum_{i=0}^{2^j-1} p_{2^j+i} D_{2^j+i}(t) + \sum_{i=0}^{k} p_{2^m+i} D_{2^m+i}(t)$$

$$= \sum_{j=0}^{m-1} \sum_{i=0}^{2^j-1} p_{2^j+i} (D_{2^j+i}(t) - D_{2^j+i+1}(t))$$

$$+ \sum_{j=0}^{m-1} D_{2^{j+1}}(t) \sum_{i=0}^{2^j-1} p_{2^j+i} + \sum_{i=0}^{k} p_{2^m+i} D_{2^m+i}(t).$$

We will make use of formula (3.4) of [3]:
\[ D_{2^j+i}(t) - D_{2^j+i}(t) = r_j(t)w_{2^j-1}(t)D_{2^j-1}(t), \quad 0 \leq i < 2^j, \]

and the formula in line 4 from below of [4, p. 46]:
\[ D_{2^m+i}(t) = D_{2^m}(t) + r_mD_i(t), \quad 1 \leq i \leq 2^m. \]

Substituting these into (3.3) yields
\[
P_nL_n(t) = -\sum_{j=0}^{m-1} r_j(t)w_{2^j-1}(t)\sum_{i=0}^{2^j-1} p_{2^j+i}D_{2^j-i}(t)
+ \sum_{i=0}^{m-1} (P_{2^i+1} - P_{2^i-1})D_{2^i-1}(t)
+ (P_n - P_{n-k-1})D_{2^m}(t) + r_m(t)\sum_{i=1}^{k} p_{2^m+i}D_i(t).
\]

Hence (3.2) follows, by noting that
\[ D_i(t) = iK_i(t) - (i - 1)K_{i-1}(t), \quad i \geq 1, \quad K_0(t) := 0, \]
(see (3.1)) and accordingly
\[
\sum_{i=0}^{2^j-1} p_{2^j+i}D_{2^j-i}(t) = \sum_{i=1}^{2^j} p_{2^j+1-i}D_i(t)
= \sum_{i=1}^{2^j-1} i(p_{2^j+1-i} - p_{2^j+1-i-1})K_i(t) + 2^j p_{2^j}K_{2^j}(t).
\]

Motivated by (3.2), we define a linear operator \( R_n \) by
\[
R_n(t) := \frac{1}{P_n}\sum_{i=1}^{k} p_{2^m+i}D_i(t),
\]
where \( n := 2^m + k, 1 \leq k \leq 2^m \), and \( m \geq 1 \). A Sidon type inequality of [2] implies that \( R_n \) as well as the weighted mean kernel \( L_n \) defined in (2.3) are quasi-positive.

**LEMMA 3.** Let \( \{p_k\} \) be a sequence of nonnegative numbers either nondecreasing and satisfying condition (2.4) or merely nonincreasing, and let \( R_n \) be defined by (3.4). Then there exists a constant \( C \) such that
\[
I_n := \int_0^1 |R_n(t)|dt \leq C, \quad n \geq 1. \tag{3.5}
\]

**PROOF.** By [2, Lemma 1 for \( p = 2 \)],
\[
I_n \leq \frac{4k^{1/2}}{P_n}\left(\sum_{i=1}^{k} p_{2^m+i}\right)^{1/2}
\]
Due to monotonicity,
\[
I_n \leq \begin{cases} \frac{4kp_n}{P_n} \leq \frac{2np_n}{P_n} & \text{if } \{p_k\} \text{ is nondecreasing,} \\ \frac{4k^2p_{2^m+1}}{P_n} \leq 4 & \text{if } \{p_k\} \text{ is nonincreasing.} \end{cases}
\]
By (2.4), hence (3.5) follows.
LEMMA 4 (see [3]). If \( g \in \mathcal{P}_2^m, f \in L^p \), where \( m \geq 0 \) and \( 1 \leq p \leq \infty \), then

\[
\left\| \int_0^1 r_m(t)g(t)[f(\cdot + t) - f(\cdot)]dt \right\|_p \leq 2^{-1} \omega_p(f, 2^{-m})\|g\|_1.
\]

4. PROOFS OF THEOREMS 1-3.

PROOF OF THEOREM 1. We shall present the details only for \( 1 \leq p < \infty \). By (2.2), (3.2), and the usual Minkowski inequality,

\[
P_n\|t_n(f) - f\|_p = \left\{ \int_0^1 \left| \int_0^1 P_n L_n(t)[f(x+t) - f(x)]dt \right|^p dx \right\}^{1/p} \leq \sum_{j=0}^{m-1} \left\{ \int_0^1 \left| \int_0^1 r_j(t)g_j(t)[f(x+t) - f(x)]dt \right|^p dx \right\}^{1/p} + \sum_{j=0}^{m-1} \left\{ \int_0^1 \left| \int_0^1 r_j(t)h_j(t)[f(x+t) - f(x)]dt \right|^p dx \right\}^{1/p}
\]

\[
+ \sum_{j=0}^{m-1} (P_{2j+1} - P_{2j}) \left\{ \int_0^1 \left| \int_0^1 D_{2j+1}(t)[f(x+t) - f(x)]dt \right|^p dx \right\}^{1/p} + (P_n - P_{n-k-1}) \left\{ \int_0^1 \left| \int_0^1 D_{2m}(t)[f(x+t) - f(x)]dt \right|^p dx \right\}^{1/p} + P_n \left\{ \int_0^1 \left| \int_0^1 r_m(t)R_n(t)[f(x+t) - f(x)]dt \right|^p dx \right\}^{1/p}
\]

\[
=: I_{1n} + I_{2n} + I_{3n} + I_{4n} + I_{5n}, \quad \text{say, (4.1)}
\]

where

\[
g_j(t) := w_{2j-1}(t) \sum_{i=1}^{2j-1} i(p_{2j+1,i} - p_{2j+1,i+1})K_i(t),
\]

\[
h_j(t) := 2^j p_{2j} w_{2j-1}(t)K_j(t).
\]

From Lemma 1,

\[
\int_0^1 |g_j(t)|dt \leq \sum_{i=1}^{2j-1} |i|p_{2j+1,i} - p_{2j+1,i+1}| \int_0^1 |K_i(t)|dt \leq 2 \sum_{r=2^{j+1}}^{2^{j+1}+1} (2^{j+1} - r)|p_r - p_{r-1}| =: A_j, \quad \text{say},
\]

If \( \{p_k\} \) is nondecreasing, we have

\[
A_j = 2^{j+2} \sum_{r=2^{j+1}}^{2^{j+1}+1} (p_r - p_{r-1}) - 2 \sum_{r=2^{j+1}}^{2^{j+1}+1} (rp_r - (r-1)p_{r-1}) + 2 \sum_{r=2^{j+1}}^{2^{j+1}+1} p_{r-1} = 2^{j+2}(p_{2j+1} - p_{2j}) - 2((2^{j+1} - 1)p_{2j+1} - 2^{j+1}p_{2j}) + 2(P_{2j+1} - P_{2j}) < 2(P_{2j+1} - P_{2j}) \leq 2^{j+1}p_{2j+1}.
\]

If \( \{p_k\} \) is nonincreasing, we have

\[
A_j = 2^{j+2} \sum_{r=2^{j+1}}^{2^{j+1}+1} (p_{r-1} - p_r) + 2 \sum_{r=2^{j+1}}^{2^{j+1}+1} (rp_r - (r-1)p_{r-1}) - 2 \sum_{r=2^{j+1}}^{2^{j+1}+1} p_{r-1} < 2^{j+1}p_{2j}.
\]
Thus, by Lemma 4, for \( \{p_k\} \) nondecreasing,
\[
I_{1n} \leq \sum_{j=0}^{m-1} 2^j p_{2^j+1-1} \omega_p(f, 2^{-j}),
\tag{4.2}
\]
and for \( \{p_k\} \) nonincreasing,
\[
I_{1n} \leq \sum_{j=0}^{m-1} 2^j p_{2^j} \omega_p(f, 2^{-j}).
\tag{4.3}
\]

Again, by Lemmas 1 and 4,
\[
\int_0^1 |h_j(t)| dt \leq 2^j p_{2^j} \int_0^1 K_{2^j}(t) dt = 2^j p_{2^j},
\]
whence
\[
I_{2n} \leq 2^{-1} \sum_{j=0}^{m-1} 2^j p_{2^j} \omega_p(f, 2^{-j}).
\tag{4.4}
\]

Since
\[
D_{2^m}(t) = \begin{cases} 
2^m & \text{if } t \in [0, 2^{-m}), \\
0 & \text{if } t \in [2^{-m}, 1)
\end{cases}
\]
(see, e.g., [5, p.7]), by the generalized Minkowski inequality,
\[
I_{3n} \leq \sum_{j=0}^{m-1} (P_{2^j+1-1} - P_{2^j-1}) \int_0^1 D_{2^j+1}(t) \left( \int_0^1 |f(x+t) - f(x)|^p dx \right)^{1/p} dt
\tag{4.5}
\]
\[
\leq \sum_{j=0}^{m-1} (P_{2^j+1-1} - P_{2^j-1}) \omega_p(f, 2^{-j})
\]
and
\[
I_{4n} \leq (P_n - P_{n-k-1}) \omega_p(f, 2^{-m}).
\tag{4.6}
\]

Note that
\[
P_{2^j+1-1} - P_{2^j-1} \leq \begin{cases} 
2^j p_{2^j+1-1} & \text{if } \{p_k\} \text{ is nondecreasing,} \\
2^j p_{2^j} & \text{if } \{p_k\} \text{ is nonincreasing.}
\end{cases}
\tag{4.7}
\]

By Lemmas 3 and 4,
\[
I_{5n} \leq 2^{-1} P_n \omega_p(f, 2^{-m}) \int_0^1 |R_n(t)| dt
\tag{4.8}
\]
\[
\leq 2^{-1} C P_n \omega_p(f, 2^{-m}).
\]

Combining (4.1) – (4.8) yields (2.5) and (2.6).

PROOF OF THEOREM 2. For \( \{p_k\} \) nondecreasing we have, from (2.4) and (2.5),
\[
\|t_n(f) - f\|_p = O\left( \frac{P_n}{P_n} \sum_{j=0}^{m-1} 2^{(1-\alpha)j} + 2^{-am} \right)
\]
\[
= O\left( 2^{-m} \sum_{j=0}^{m-1} 2^{(1-\alpha)j} + 2^{-am} \right).
\]

Hence (2.7) follows easily.
For \( \{p_k\} \) nonincreasing, (2.8) is immediate.

**PROOF OF THEOREM 3.** By a theorem of Watari [6]

\[ \|s_{2m}(f) - f\|_p \leq 2E_{2m}(f, L^p). \]

Thus, from (2.9),

\[ \|s_{2m}(f) - f\|_p = o(P_{2m}^{-1}). \]  \hspace{1cm} (4.9)

Clearly,

\[ P_{2m}\{s_{2m}(f, x) - t_{2m}(f, x)\} = \sum_{k=1}^{2^m} p_k\{s_{2m}(f, x) - s_k(f, x)\} \]

\[ = \sum_{k=1}^{2^m-1} p_k\sum_{i=k}^{2^m-1} a_iw_i(x) \]

\[ = \sum_{i=1}^{2^m-1} P_ia_iw_i(x). \]

Now (2.9) and (4.9) imply

\[ \lim_{m \to \infty} \left\| \sum_{i=1}^{2^m-1} P_ia_iw_i(x) \right\|_p = 0. \]

Since the \( L^p \)-norm dominates the \( L^1 \)-norm for \( p > 1 \), it follows that for \( j \geq 1 \),

\[ |P_ja_j| = \lim_{m \to \infty} \left| \int_0^1 w_j(x) \sum_{i=1}^{2^m-1} P_ia_iw_i(x) dx \right| \]

\[ \leq \lim_{m \to \infty} \left\| \sum_{i=1}^{2^m-1} P_ia_iw_i(x) \right\|_1 = 0. \]

Hence we conclude that \( a_j = 0 \) for all \( j \geq 1 \). Therefore, \( f = a_0 \), a constant.

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