ABSTRACT. We study some conditions on the Ricci tensor of real hypersurfaces of quaternionic projective space obtaining among other results an improvement of the main theorem in [9].

KEY WORDS AND PHRASES. Quaternionic projective space, real hypersurface, Ricci tensor.


1. INTRODUCTION.

Let $M$ be a real hypersurface, which in the following we shall always consider connected, of a quaternionic projective space $QP^m$, $m \geq 2$, with metric $g$ of constant quaternionic sectional curvature $4$. Let $\zeta$ be the unit normal vector field on $M$ and $\{J_1, J_2, J_3\}$ a local basis of the quaternionic structure of $QP^m$, see [2]. Then $U_i = -J_i\zeta$, $i = 1, 2, 3$, are tangent to $M$. Let $S$ be the Ricci tensor of $M$.

In [6] we studied pseudo-Einstein real hypersurfaces of $QP^m$. These are real hypersurfaces satisfying

$$qX + aU_i + b_i \zeta = b_i \zeta, \quad (1.1)$$

for any $X$ tangent to $M$, where $a$ and $b$ are constant. If $m \geq 3$ we obtained that $M$ is pseudo-Einstein if it is an open subset of either a geodesic hypersphere or of a tube of radius $r$ over $QP^k$, $0 < k < m - 1, 0 < r < \Pi/2$ and $\cot^2 r = (4k + 2)/(4m - 4k - 2)$.

As a corollary we also obtained that the unique Einstein real hypersurfaces of $QP^m, m \geq 2$, are open subsets of geodesic hyperspheres of $QP^m$ of radius $r$ such that $\cot^2 r = 1/2m$.

The purpose of the present paper is to study several conditions on the Ricci tensor of $M$. Concretely in 3 we prove the following result: if $X$ is tangent to $M$ we shall write $J_iX = \Phi_iX + f_i(X)\zeta$, $i = 1, 2, 3$, where $\Phi_iX$ denotes the tangent component of $J_iX$ and $f_i(X) = g(X, U_i)$. Then

THEOREM 1. Let $M$ be a real hypersurface of $QP^m, m \geq 3$, such that $\Phi_iS = S\zeta$, $i = 1, 2, 3$. Then $M$ is an open subset of a tube of radius $r, 0 < r < \Pi/2$, over $QP^k, k \in \{0, \ldots, m - 1\}$.

This theorem generalizes results obtained by Pak in [7].

In [9] we studied real hypersurfaces of $QP^m$ with harmonic curvature for which $U_i, i = 1, 2, 3$, are eigenvectors of the Weingarten endomorphism of $M$ with the same principal curvature. A real hypersurface has harmonic curvature if

$$\nabla_X S = \nabla_X S = \nabla_Y S = \nabla_X S$$

(1.2)
for any \( X, Y \) tangent to \( M \), where \( \nabla \) denotes the covariant differentiation of \( M \). In 4 we shall improve the result of [9] showing that the condition about principality of \( U_i, i = 1, 2, 3 \), is unnecessary. Concretely we obtain

**THEOREM 2.** A real hypersurface of \( \mathbb{P}^m \), \( m \geq 2 \), has harmonic curvature if and only if it is Einstein.

As a consequence we can classify Ricci-parallel real hypersurfaces of \( \mathbb{P}^m \), that is, real hypersurfaces such that \( \nabla X S = 0 \) for any \( X \) tangent to \( M \). We get

**COROLLARY 3.** The unique Ricci-parallel real hypersurfaces of \( \mathbb{P}^m \), \( m \geq 2 \), are open subsets of geodesic hyperspheres of radius \( r \), \( 0 < r < \pi/2 \), such that \( \cot^2 r = 1/2m \).

From this result we introduce in 5 a condition that generalize Ricci-parallel real hypersurfaces. We shall say that a real hypersurface of \( \mathbb{P}^m \) is pseudo Ricci-parallel if it satisfies

\[
(\nabla X S)Y = c \sum_{i=1}^{3} \{g(\Phi_i X, Y)U_i + f_i(Y)\Phi_i X\}
\]

for any \( X, Y \) tangent to \( M \), \( c \) being a nonnull constant. We obtain

**THEOREM 4.** \( M \) is a pseudo Ricci-parallel real hypersurface of \( \mathbb{P}^m \), \( m \geq 2 \), if and only if it is an open subset of a geodesic hypersphere.

Finally, we characterize pseudo-Einstein real hypersurfaces of \( \mathbb{P}^m \) by the following

**THEOREM 5.** Let \( M \) be a real hypersurface of \( \mathbb{P}^m \), \( m \geq 3 \), then

\[
\|S\|^2 \geq \sum_{i=1}^{3} (f_i(SU_i))^2 + (\rho - \sum_{i=1}^{3} f_i(SU_i))^2)/4(m-1)
\]

where \( \rho \) denotes the scalar curvature of \( M \). The equality holds if and only if \( M \) is pseudo-Einstein.

2. **PRELIMINARIES.**

Let us call \( D = \text{Span}\{U_1, U_2, U_3\} \) and \( D \) its orthogonal complement in \( TM \). Let \( X, Y \) be vector fields tangent to \( M \). Then, [6], we have

\[
\Phi_i^2 X = -X + f_i(X)U_i
\]

\[
g(\Phi_i X, Y) + g(X, \Phi_i Y) = 0, \Phi_i U_i = 0, \Phi_j U_k = -\Phi_k U_j = U_i
\]

where \( i = 1, 2, 3 \) and \((j, k, t)\) is a circular permutation of \((1,2,3)\).

From the expression of the curvature tensor of \( \mathbb{P}^m \), [2], the Ricci tensor of \( M \) is given by

\[
SX = (4m + 7)X - 3\sum_{i=1}^{3} f_i(X)U_i + hAX - A^2 X
\]

for any \( X \) tangent to \( M \), where \( h = \text{trace}(A) \). Moreover, [6],

\[
\nabla X U_i = q_k(X)U_j - q_j(X)U_k + \phi_i AX
\]

for any \( X \) tangent to \( M \), \((i, j, k)\) being a circular permutation of \((1,2,3)\) and \( q_i, i = 1, 2, 3 \), certain local 1-forms on \( M \) (see [2]). Finally the equation of Codazzi is given by

\[
(\nabla X A)Y - (\nabla Y A)X = \sum_{i=1}^{3} \{f_i(X)\Phi_i Y - f_i(Y)\Phi_i X + 2g(X, \Phi_i Y)U_i\}
\]

for any \( X, Y \) tangent to \( M \).
3. PROOF OF THEOREM 1.

Let us call $H = A^2 - fA$, $f$ being a differentiable function on $M$

If we suppose that $H \Phi_i = \Phi_i H$, $i = 1, 2, 3$, from (2.2) $H \Phi_1 U_1 = 0 = \Phi_1 H U_1$. This implies $0 = \Phi_i^2 H U_1 = -H U_1 + f_i (H U_1) U_1$. That is, $U_1$ is an eigenvector of $H$. Similarly, $U_2$ and $U_3$ are also eigenvectors of $H$. Let us consider $T, M = H(\alpha_1) \oplus H(\alpha_2) \oplus \cdots \oplus H(\alpha_p)$, where $H(\alpha_j) = \{ X \in T, M \mid H X = \alpha_j X \}$. Suppose that $U_i \in H(\alpha_i), i = 1, 2, 3$.

If $X \in \mathbb{D}$ is such that $X \in H(\alpha_i), H \Phi_i X = \Phi_i H X = \alpha_i \Phi_i X$, that is, $\Phi_i X \in H(\alpha_i), j = 1, 2, 3$. Moreover $H \Phi_i U_j = \Phi_i H U_j = \alpha_i \Phi_i U_j, j = 1, 2, 3$. If $j = 2$, we obtain that $H U_3 = \alpha_1 U_3$. If $j = 3$ we obtain $H U_2 = \alpha_1 U_2$. Thus $\alpha_1 = \alpha_2 = \alpha_3$. Then $H(\alpha_1)$ is odd-dimensional and from (2.5) the proof of Theorem 6 in [6] implies that $U_i, i = 1, 2, 3$, are eigenvectors of $A$.

If we now consider a real hypersurface of $Q_{P^m}, m \geq 3$, such that $\Phi_i H = H \Phi_i, i = 1, 2, 3$, for $f = -h$. Thus $U_i, i = 1, 2, 3$, are eigenvectors of $A$. Thus, [1], $M$ is an open subset of either a tube of radius $\tau$, $0 < \tau < \Pi/2$, over $Q_{P^k}$, $k = \{0, \ldots, m-1\}$ or of a tube of radius $\tau, 0 < \tau < \Pi/4$, over $C_{P^m}$

Let us consider the second case. The eigenvalues of $A$ are $\cot(\tau)$ with multiplicity $2(m-1)$, $-\tan(\tau)$ with multiplicity $2(m-1)$, $2\cot(2\tau)$ with multiplicity 1, and $-2\tan(2\tau)$ with multiplicity 2. Let $X$ be a unit vector field such that $AX = \cot(\tau) X$. Then $\Phi_2 S X = (4m + 7 + h\cot(\tau) - \cot^2(\tau))\Phi_2 X$ and $S \Phi_2 X = (4m + 7 - h\tan(\tau) - \tan^2(\tau))\Phi_2 X$. From this we have $h(\cot(\tau) + \tan(\tau)) + \tan^2(\tau) - \cot^2(\tau) = 0$. Thus either $\cot(\tau) + \tan(\tau) = 0$ and this implies $\cot^2(\tau) = -1$ which is impossible or $h + \tan(\tau) - \cot(\tau) = 0$. As $h = 2(m-1)(\cot(\tau) - \tan(\tau)) + 2\cot(2\tau) - 4\tan(2\tau)$, it is easy to see that $\tan^2(2\tau) = m - 1$.

On the other hand, $\Phi_2 S U_1 = (4m + 4 + 2h\cot(2\tau) - 4\cot^2(2\tau)) U_3$ and $S \Phi_2 U_1 = -S U_3 = 4m + 4 - 2h\tan(2\tau) - 4\tan^2(2\tau) U_3$. This implies $h(\cot(2\tau) + \tan(2\tau)) - 2(\cot^2(2\tau) - \tan^2(2\tau)) = 0$. Thus either $\cot(2\tau) + \tan(2\tau) = 0$ which implies $\cot^2(2\tau) = -1$ which is impossible or $h - 2(\cot(2\tau) - \tan(2\tau)) = 0$. This implies $\tan^2(2\tau) = 2(m-1)$. Thus $m - 1 = 2(m-1)$. Then $m = 1$ which is impossible. This finishes the proof.

4. PROOF OF THEOREM 2.

As $M$ has harmonic curvature for any $X, Y$ tangent to $M$ we get

$$\nabla_X SY - \nabla_Y SX = S([X, Y]) \quad (4.1)$$

Then for any $X, Y, Z$ tangent to $M$ we obtain

$$R(Z, X) SY = \nabla_Z (\nabla_X SY - \nabla_X \nabla_Y Z) =$$

$$= S(R(Z, X) Y) + \nabla_Z (\nabla_Y SX + \nabla_X SY) - \nabla_Y (\nabla_X SY) - \nabla_X (\nabla_Y SX) \quad (4.2)$$

where $R$ denotes the curvature tensor of $M$.

From (4.2), (1.2) and the first identity of Bianchi we get

$$\sigma(R(X, Y) SZ) = 0 \quad (4.3)$$

for any $X, Y, Z$ tangent to $M$, where $\sigma$ denotes the cyclic sum. The result now follows from the main theorem of [8].
5. PROOFS OF THEOREMS 4 AND 5.

Firstly, let us suppose that $M$ is pseudo Ricci-parallel. Then applying (1.3) and (2.4) we have
\[\nabla_w (\nabla_x S) Y - (\nabla_{x W} S) Y = c\Sigma_{i=1}^3 \{g(\Phi, X, Y)\Phi, AW + g(Y, \Phi, AW)\Phi, X + f_i(X)g(AW, Y)U_i + f_i(Y)g(X, AW)U_i \}
\]
for any $X, Y, W$ tangent to $M$. If in (5.1) we exchange $X$ and $W$ we get
\[\nabla_w (\nabla_x S) Y - (\nabla_{y W} S) Y = c\Sigma_{i=1}^3 \{g(\Phi, Y, X)\Phi, AW + g(X, \Phi, AW)\Phi, Y + f_i(Y)g(AW, X)U_i + f_i(X)g(Y, AW)U_i \}
\]
Taking a local orthonormal frame $\{E_1, \ldots, E_{4m-1}\}$ of $TM$, from (5.2), (2.1) and (2.2) we have
\[\Sigma_{i=1}^{4m-1} g((R(E_i, X)S)Y, E_j) = c\Sigma_{i=1}^3 \{f_i(X)g(AW, Y)U_i + f_i(Y)g(X, AW)U_i \}
\]
Now the left hand side of (5.3) is symmetric with respect to $X, Y$ (see [4]). Thus (5.3) gives
\[3c\Sigma_{i=1}^3 f_i(X)g(AW, Y)U_i = 3c\Sigma_{i=1}^3 f_i(Y)g(X, AW)U_i
\]
for any $X, Y$ tangent to $M$.

We know, [1], that if $g(AD, \Omega) = \{0\} \cup \{1, 2, 3\}$ are principal for $A$. Let us suppose that $g(AD, \Omega) = \{0\} \cup \{1, 2, 3\}$. We shall distinguish the following cases where $\Omega$ denotes the $\Omega$-component of $X$.

(i) $(AU_1)^\Omega = (AU_2)^\Omega = 0$ and $(AU_3)^\Omega \neq 0$. Then we write $AU_1 = \alpha_1 X_1 + \beta_1 Y_1$ where $X_1 \in \Omega$ and $Y_1 \in \Omega^\perp$ are unit. If we take in (5.5) $X = X_1$ and $Y = U_1$ we have $0 = \Sigma_{i=1}^3 f_i(Y)g(X, X_1) = g(\Phi, X_1, X_1) = \alpha$. Then $g(AD, \Omega^\perp) = \{0\}$.

(ii) $(AU_1)^\Omega = 0$ and $(AU_2)^\Omega$ are linearly dependent. We write $AU_1 = \alpha_1 X_1 + \beta_1 U_1 + \beta_2 U_2 + \beta_3 U_3$ and $AU_2 = \alpha_2 X_1 + \beta_2 U_2 + \beta_3 U_3$ where $X_1 \in \Omega$ and $Y_1 \in \Omega^\perp$ are unit. If we take in (5.5) $X = X_1$, $Y = U_1$ we obtain $0 = g(\Phi, X_1, Y_1) = \alpha_1$. Now we have case (i).

It is easy to see that the rest of cases (if $(AU_3)^\Omega = 0$ and $(AU_1)^\Omega$, $(AU_2)^\Omega$ are linear independent or if $(AU_i)^\Omega \neq 0, i = 1, 2, 3$) are similar. That is, $g(AD, \Omega^\perp) = \{0\}$. Thus $M$, [1], is open subset of a geodesic hypersphere or of a tube of radius $r$, $0 < r < \Pi/2$, over $QP_k$, $k \in \{1, \ldots, m-2\}$ or of a tube of radius $r$, $0 < r < \Pi/4$, over $CP^m$.

In the second case, $M$ has 3 distinct principal curvatures $\lambda_1 = \cot(r)$ with multiplicity $4(m-k-1)$, $\lambda_2 = -\tan(r)$ with multiplicity $4k$ and $\lambda_3 = 2\cot(2r)$ with multiplicity $3$.

Let us take a unit $X$ such that $AX = \lambda_1 X$. If we develop $g((\nabla_x S)\Phi, X, U_1)$ we obtain $c = -(\cot(r) - \cot^2(r) + 3 - 2\cot^2(2r) + 4\cot^2(2r))\cot(r)$. If we take a unit $Y$ such that $AY = \lambda_2 Y$ and develop $g((\nabla_y S)\Phi, Y, U_1)$ we get $c = (\cot(r) - \tan(r) + 3 - 2\cot^2(2r) + 4\cot^2(2r))\tan(r)$. From this we get $\tan^2(r) = -1$ which is impossible.

The same result is obtained if $M$ is an open subset of a tube of radius $r$, $0 < r < \Pi/4$, over $CP^m$. 
On the other hand, if $M$ is an open subset of a geodesic hypersphere $M$ has two distinct principal curvatures, $\lambda = \cot(r)$ with multiplicity $4(m - 1)$ and $\alpha = 2\cot(2r)$ with multiplicity 3. Then it is easy to see that such an $M$ satisfies (1.3) and this finishes the proof.

Finally, the fact of a real hypersurface $M$ of $QP^{n}$, $m \geq 3$, being pseudo-Einstein is equivalent to the fact that $g(SX, Y) = ag(X, Y)$ for any $X, Y \in \mathfrak{D}$ and that $U_{i}, i = 1, 2, 3$, are eigenvectors of $S$. This is equivalent to $\sum_{i=1}^{3}f_{i}(X)SU_{i} - \rho_{0}X - \rho_{0}g(SU_{i}, U_{i})$. This is equivalent to $S = \sum_{i=1}^{3}f_{i}(X)SU_{i} + \rho_{0}\sum_{i=1}^{3}f_{i}(X)U_{i} + \rho_{0}g(SU_{i}, U_{i})$. If we define the tensor $P$ as $P(X, Y) = g(SX, Y) - \rho_{0}g(X, Y) + \rho_{0}\sum_{i=1}^{3}f_{i}(X)f_{i}(Y) + \sum_{i=1}^{3}(f_{i}(SU_{i}), f_{i}(X)f_{i}(Y))f_{i}(SY) - f_{i}(SX)f_{i}(Y)$ for any $X, Y$ tangent to $M$ and compute its length we obtain

$$\|P\|^2 = \|S\|^2 - 4(m - 1)\rho_{0} - 2\sum_{i=1}^{3}SU_{i}\|^2 + \sum_{i=1}^{3}(f_{i}(SU_{i}))^2$$

(5.6)

But it is easy to see that for any real hypersurface $M$

$$\sum_{i=1}^{3}g(SU_{i}, SU_{i}) \geq \sum_{i=1}^{3}(g(SU_{i}, U_{i}))^2$$

(5.7)

Then (1.4) follows from (5.6), (5.7) and the expression of $\rho_{0}$. Moreover if $U_{i}$, $i = 1, 2, 3$, are eigenvectors of $S$ we obtain the equality in (1.4). Thus we have finished the proof of Theorem 5.

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