EXISTENCE OF SOLUTIONS FOR A NONLINEAR HYPERBOLIC-PARABOLIC EQUATION IN A NON-CYLINDER DOMAIN

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ABSTRACT. In this paper, we study the existence of global weak solutions for the equation

\[ k_2(x)u'' + k_1(x)u' + A(t)u + u \int u f(t) = 0 \]

in the non-cylinder domain \( Q \) in \( \mathbb{R}^{n+1} \); \( k_1 \) and \( k_2 \) are bounded real functions, \( A(t) \) is the symmetric operator \( a_{ij}(x,t) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x,t) \frac{\partial}{\partial x_j} \right) \)

where \( a_{ij} \) and \( f \) are real functions given in \( Q \). For the proof of existence of global weak solutions we use the Faedo-Galerkin method, compactness arguments and penalization.

KEY WORDS AND PHRASES. Existence of weak solutions, Faedo-Galerkin method, compactness arguments.

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INTRODUCTION AND TERMINOLOGY.

Let \( T \geq 0 \) be a positive real number, \( O \) a bounded open set of \( \mathbb{R}^n \) and \( Q \subset O \times [0,T) \) a non-cylindrical domain in \( \mathbb{R}^{n+1} \).

In the cylinder \( \Omega \times (0,T) \), where \( \Omega \subset \mathbb{R}^n \) is a bounded open set, Bensoussan et al. [1] and Lions [7] have studied the homogenization for the following Cauchy problem:

\[ k_2(x)u'' + k_1(x)u' + \Delta u = f \text{ in } \Omega \]
\[ u(x,0) = u_0(x) \text{ and } k_2(x)u'(x,0) = k_1^{1/2}(x)u(x), x \in \Omega \]

Many authors have been investigating the solvability of solution for the nonlinear equations associated with problem (I) see: Larkin [4], Lima [5], Medeiros [9], Medeiros [10], Medeiros [11], Melo [12], Maciel [13], Neves [14] and Vagrov [16].

In the non-cylindrical domain \( Q \), Lions, J.L. [8] studied the existence and uniqueness of global weak solutions for nonlinear equations associated with problem (II) with nonlinearity of type \( |u|^{\alpha}u \).

Let \( \Omega_t = Q \cap \{ t = s \} \) be a plane in \( \mathbb{R}^{n+1} \). Analogously \( \Omega_0 = Q \cap \{ t = 0 \} \) and \( \Omega_T = Q \cap \{ t = T \} \); \( \partial Q = \Gamma \) the boundary of \( Q \); \( \Gamma_s = \partial Q \cap \{ t = s \} \) the boundary of \( \Omega_s \) and \( \Sigma = \bigcup_{0 \leq s \leq T} \Gamma_s \) lateral boundary of \( Q \). Therefore \( Q \) is a subset of \( O \times (0,T) \) whose boundary is \( \Omega_0 \cap \Sigma \cap \Omega_T \).

Let’s denote by \( \langle \cdot, \cdot \rangle \) and \( | \cdot | \) the inner product and the norm in \( L^2(\Omega) \) and by \( (\cdot, \cdot) \) and
We use Faedo-Galerkin’s method and compactness arguments, see Lions, J.L. [7]

1. Assumptions and main result.

If we assume the following hypothesis:

(H.1) Let \( \Omega_t^* \) be the projection of the \( \Omega_t \) on the hyperplane \( t = 0 \). We may assume \( \Omega_t^* \subseteq \Omega_t^* \) if \( t \leq s \).

(H.2) For each \( t \in [0, T] \), \( \Omega_t \) has the following regularity: If \( u \in H^1_0(\Omega) \) and \( u = 0 \) a.e. on \( \Omega \), then the restriction of \( u \) to \( \Omega_t \) belongs to \( H^1_0(\Omega_t) \).

On the functions \( k_1, k_2 \) and \( a_{ij} \) we take:

(H.3) \( k_1, k_2 \in L^\infty(\Omega_t); k_1(x) \geq \beta > 0, \beta \in \mathbb{R}; k_2(x) \geq 0 \) for each \( t \in [0, T] \).

(H.4) \( a_{ij} = a_{ji} \in L^\infty(\Omega \times (0, T)) \) and \( a_{ij}' = \frac{\partial}{\partial x_j} a_{ij} \in L^\infty(\Omega \times (0, T)) \).

There is \( 0 < \delta \in \mathbb{R} \) such that

\[
\sum_{i,j=1}^{n} a_{ij}(x,t) \xi_i \xi_j \geq \delta (|\xi_1|^2 + \cdots + |\xi_n|^2), \quad (x,t) \in \Omega \times (0, T), \quad \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n
\]

Let \( a(t, u, v) \) denote the bilinear form associated to the operator \( A(t) \). From (H.4) and, using Cauchy-Schwartz, we obtain:

\[
a(t, u, v) \leq C \| u \| \cdot \| v \|, \quad \forall u, v \in H^1_0(\Omega);
\]

Also by Poincaré-Friedrichs inequality and of (H.4), there exists \( \alpha > 0 \), real, such that:

\[
a(t, u, v) \geq \alpha \| u \|^2, \quad \forall u \in H^1_0(\Omega)
\]

Therefore, from the above inequalities, we conclude that \( a(t, \cdot, \cdot) \) is continuous and coercive in \( H^1_0(\Omega) \times H^1_0(\Omega) \).

Now let's consider the main result.

**Theorem 1.** Suppose the hypothesis (H.1)-(H.4) are satisfied and that

\[
f \in L^2(\Omega) \quad (1.1)
\]

\[
u_0 \in H^1_0(\Omega_0) \quad (1.2)
\]

\[
u_1 \in L^2(\Omega_0) \text{ are given, with } 0 < \rho \leq \frac{4}{n-2} \quad (1.3)
\]

Then there exists a function \( u: \Omega \to \mathbb{R} \) such that

\[
u \in L^\infty(0, T; H^1_0(\Omega_t)) \quad (1.4)
\]

\[
u' \in L^\infty(0, T; L^2(\Omega_t)), \quad \sqrt{k_2(x)} \nu' \in L^\infty(0, T; L^2(\Omega_t)) \quad (1.5)
\]

\[
k_2(x) \nu'' \in L^p(0, T; H^{-1}(\Omega_t)) \text{ with } \frac{1}{p} + \frac{1}{p'} = 1, \quad p = \rho + 2 \text{ and } \rho \quad (1.6)
\]

is a solution (1) in the weak sense in \( \Omega \), i.e.,
\[
\frac{d}{dt}(k_2(x)u_t(t), v) + (k_1(x)u_t(t), v) + a(t, u(t), v) + (|u(t)|^p u(t), v) = (f(t), v),
\]
in \(D'(0, T), \forall v \in H_0^1(\Omega_t). \)
\[u(x, 0) = u_0(x); \quad k_2(x)u_t(x, 0) = \sqrt{k_2(x)}u_1 \text{ in } \Omega_0\]

**PROOF.** The idea is to transform the non-cylindrical problem in the cylindrical problem, through the penalization function, \(M \in L^\infty(O \times (0, T))\), that was introduced by J.L. Lions [8], given by:
\[
M(x, t) = \begin{cases} 
0, & \text{in } Q \\
1, & \text{in } O \times (0, T) \setminus Q.
\end{cases}
\]

For each \(\epsilon > 0\), we will find \(U^\epsilon\) in the cylinder \(O \times (0, T)\), solution of the perturbed problem \((P_\epsilon)\) below
\[
\tilde{k}_2(x)U_{tt} + \tilde{k}_1(x)U_t + A(t)U^\epsilon + \frac{1}{\epsilon} MU^\epsilon + \|U^\epsilon\| U^\epsilon = \tilde{f}
\]
\[
U^\epsilon(0) = \tilde{u}_0
\]
\[
\tilde{k}_2U^\epsilon(0) = \sqrt{\tilde{k}_2(x)}\tilde{u}_1
\]
\[
U^\epsilon = 0 \text{ in the } \partial(O \times (0, T)) = \tilde{\Sigma}
\]
where \(\tilde{k}_2(x) = k_2(x) + \epsilon; U_t = \frac{\partial}{\partial t} U; U_{tt} = \frac{\partial^2}{\partial t^2} U; \quad \tilde{u}_0 = \begin{cases} 
0, & \text{in } \Omega_0 \\
u_0, & \text{in } O \setminus \Omega_0
\end{cases}\)

Therefore, \(\tilde{u}_0 \in H_0^1(O)\). Analogously \(\tilde{u}_1 \in L^2(O)\);
\[
\tilde{f} = \begin{cases} 
f, & \text{in } Q \\
0, & \text{in } O \times (0, T) \setminus Q
\end{cases}
\]

Therefore \(\tilde{f} \in L^2(O \times (0, T))\);
\[
\tilde{k}_1(x) = \begin{cases} 
k_1(x), & \text{in } Q \\
\beta, & \text{in } O \times (0, T) \setminus Q
\end{cases}
\]
and \(\tilde{k}_2(x) = \begin{cases} 
k_2(x), & \text{in } Q \\
0, & \text{in } O \times (0, T) \setminus Q
\end{cases}\)

So \(\tilde{k}_1\) and \(\tilde{k}_2 \in L^\infty(O \times (0, T))\).

The proof of Theorem 1 will be a consequence of the following Theorem:

**THEOREM 2.** For each \(\epsilon > 0\), there exists one function \(U^\epsilon: O \times (0, T) \rightarrow \mathbb{R}\), solution of the problem \((P_\epsilon)\), such that:
\[
U^\epsilon \in L^\infty(0, T; H_0^1(O))
\]
\[
U^\epsilon \in L^\infty(0, T; L^2(O)), \sqrt{\tilde{k}_2(x)}U^\epsilon_t \in L^\infty(0, T; L^2(O))
\]
\[
\tilde{k}_2(x)U^\epsilon_{tt} \in L^p(0, T; H^{-1}(O))
\]
with \(\frac{1}{p} + \frac{1}{p'} = 1\) and \(p = \rho + 2\)
\[
\tilde{k}_2(x)U^\epsilon_{tt} + \tilde{k}_1(x)U^\epsilon_t + A(t)U^\epsilon + \frac{1}{\epsilon} MU^\epsilon + |U^\epsilon| U^\epsilon = \tilde{f}
\]
in the weak sense in \(O \times (0, T)\).
\[
U^\epsilon(x, 0) = \tilde{u}_0(x)
\]
\[
\tilde{k}_2(x)U^\epsilon_t(x, 0) = \sqrt{\tilde{k}_2(x)}\tilde{u}_1(x)
\]
REMARK 1. The condition \( U' = 0 \) in \( \hat{\Sigma} \) is a consequence of the fact that \( U' \) in 
\( L^2(0,T; H^1_0(\Omega)) \).

REMARK 2. For the proof of Theorem 1 it is sufficient to prove that the solution \( U' \) in 
Theorem 2 converges for \( U \) in the weak sense when \( \epsilon \to 0 \) and that the restriction of \( U \) to \( Q \) satisfies all the assertions of Theorem 1.

In this part, we use a result due to W.A. Strauss see [15].

PROOF OF THEOREM 2.

(i) Approximate Problem. It will be done by the Faedo-Galerkin method. Let \( \{w_i\}_{i=1}^\infty \subset H^1(\Omega) \) be a basis of \( H^1(\Omega) \) and \( V_m \) the subspace spanned by the \( m \) first vectors
\( w_1, w_2, \ldots, w_m \). Let \( U_m' \) be the function
\[
U_m'(x,t) = \sum_{j=1}^m g_{jm}(t)w_j(x)
\]
defined by the system
\[
(\hat{k}_2(x) \frac{\partial^2}{\partial t^2} U_m'(x,t), w_j) + (\hat{k}_1(x) \frac{\partial}{\partial t} U_m'(x,t), w_j) + a(t, U_m'(x,t), w_j) \\
+ \frac{1}{2} M \left( \frac{\partial}{\partial t} U_m'(x,t), w_j \right) + (|U_m'(x,t)|^p, w_j) = (f(t), w_j), \quad \forall j = 1, \ldots, m
\]
(1.19)
\[
U_m'(0) = U_{0m} = \sum_{j=1}^m \alpha_j w_j \to u_0 \text{ strong in } H^1(\Omega)
\]
(1.20)
\[
\frac{\partial}{\partial t} U_m'(0) = U_{1m} = \sum_{j=1}^m \beta_j w_j \to \frac{u_1}{\sqrt{k_2}} \text{ strong in } L^2(\Omega)
\]
(1.21)

The system (1.19)-(1.21) satisfies the condition of Caratheodory’s theorem see [2]. Therefore it has a solution \( U_m' \) defined in \([0, t_m], \) where \( 0 < t_m \leq T \). The a priori estimates to be obtained in the following step, show, in particular, that \( t_m = T \).

(ii) A Priori Estimates. By multiplying both sides of (1.19) by \( 2g_{jm}(t) \), and adding from 
\( j = 1 \) to \( j = m \) we obtain:
\[
\frac{d}{dt} \left| \sqrt{k_2(x)} U_m'(t) \right|^2 + 2 \sqrt{k_1(x)} U_m(t) \left( \frac{1}{2} M U_m'(t) \right) + 2a(t, U_m(t), U_m'(t)) + \frac{1}{2} \int_{\Omega} M(U_m')^2 dx \\
+ \int_{\Omega} \| U_m(s) \| p U_m'(s) U_m'(s) dx = 2(f(t), U_m'(t))
\]
(1.22)
where we wrote \( U_m \) instead of \( U_m' \) and denoted by \( U_m' = \frac{\partial}{\partial t} U_m \).

REMARK 3. We have that
\[
\frac{d}{dt} a(t, U_m(t), U_m(t)) = a'(t, U_m(t), U_m(t)) + 2a(t, U_m(t), U_m'(t));
\]
where
\[
a'(t, U_m(t), U_m(t)) = \sum_{j=1}^m \int_{\Omega} \left( \frac{\partial}{\partial t} a_{ij}(x,t) \frac{\partial}{\partial x_j} U_m(t) \frac{\partial}{\partial x_j} U_m(t) dx.
\]
Therefore,
\[
2a(t, U_m(t), U_m(t)) = \frac{d}{dt} a(t, U_m(t)) - a'(t, U_m(t)).
\]

REMARK 4. We have that
\[
\frac{1}{p} \int_{\Omega} |U_m(s)|^p dx + \int_0^t \left| \frac{1}{2} \sqrt{k_1(x)} U_m'(s) \right|^p ds + a(t, U_m(t)) + \frac{2}{p} \int_{\Omega} |U_m(s)|^p dx
\]
Therefore, in the remarks (3 and 4) below, we have, integrating (1.22) from 0 to \( t \),
\[
0 < t \leq t_m, \text{ that:}
\]
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\[ \frac{2}{\beta} \int_{0}^{t} |U_{m}(s)|^{p} \, ds + \int_{0}^{t} a'(s, U_{m}(s)) \, ds + 2 \int_{0}^{t} (f(s), U'_{m}(s)) \, ds \]

**REMARK 5.** From (20), (21) and the Sobolev Immersion, \( H^1(O) \rightarrow L^p(O), \forall \ \frac{1}{p} = \frac{1}{2} - \frac{1}{n} \), we obtain:

\[ \| U_{0m} \|_{L^p(O)} \leq C. \]
\[ \| \sqrt{k_{2t}(x)} U_{1m} \|_{L^1} \leq C; \ |a(0, U_{0m})| \leq C. \]

Here, the letter \( C \) denotes different constants.

**REMARK 6.** By using (H.4), we obtain:

\[ \int_{0}^{t} \| U_{m}(s) \|^{2} \, ds \leq C \int_{0}^{t} \| U_{m}(s) \|^{2} \, ds; \]

Therefore, from the remarks (5 and 6) below, we can write (1.23) like

\[ \left| \sqrt{k_{2t}(x)} U'_{m}(t) \right|^{2} + 2 \int_{0}^{t} \left| \sqrt{k_{1}(x)} U'_{m}(s) \right|^{2} \, ds + a(t, U_{m}(t)) + \frac{2}{\beta} \int_{0}^{t} |U_{m}(s)|^{p} \, ds \]
\[ + \frac{2}{\beta} \int_{0}^{t} M(U'_{m}(s))^{2} \, ds \leq C + C \int_{0}^{t} \| U_{m}(s) \|^{2} \, ds + \lambda \int_{0}^{t} |U'_{m}(s)|^{2} \, ds \]

(1.24)

From (1.24), if we choose \( \lambda = \beta > 0 \) (the \( \beta > 0 \) of H.3) we obtain:

\[ \int_{0}^{T} |U'_{m}(s)|^{2} \, ds \leq C + C \int_{0}^{t} \| U_{m}(s) \|^{2} \, ds, \]

(1.25)

and

\[ a(t, U_{m}(t)) \leq C + C \int_{0}^{t} \| U_{m}(s) \|^{2} \, ds + \beta \int_{0}^{t} |U'_{m}(s)|^{2} \, ds \]

(1.26)

Being \( a(t, u, v) \) coercive, we obtain from (1.25) and (1.26), that:

\[ \| U_{m}(t) \|^{2} \leq C + C \int_{0}^{t} \| u_{m}(s) \|^{2} \, ds, \quad \forall t \in [0, t_{cm}). \]

(1.27)

Gronwall's inequality implies that

\[ \| U_{m}^{\e} \| \leq C, \quad \forall m \in \mathbb{N}, \ \forall \varepsilon > 0, \ \forall t \in [0, t_{cm}). \]

(1.28)

Returning to (1.25) we obtain:

\[ \int_{0}^{t} \left\| \frac{\partial}{\partial s} U_{m}^{\e}(s) \right\|^{2} \, ds \leq C, \]

\( \forall m \in \mathbb{N}, \ \forall \varepsilon > 0, \ \forall t \in [0, t_{cm}). \)

(1.29)

The priori estimative (1.24) shows that \( t_{cm} = T \). Therefore,

\[ \left| \sqrt{k_{2t}(x)} \frac{\partial}{\partial s} U_{m}(t) \right|^{2} + 2 \int_{0}^{t} \left| \sqrt{k_{1}(x)} \frac{\partial}{\partial s} U_{m}(s) \right|^{2} \, ds + a(t, U'_{m}(t)) \]
\[ + \frac{2}{\beta} \int_{0}^{t} |U'_{m}(s)|^{p} \, ds + \frac{2}{\beta} \int_{0}^{t} M(U'_{m}(s))^{2} \, ds \leq C \]

\( \forall m \in \mathbb{N}, \forall \varepsilon > 0 \) and \( \forall t \in [0, T]. \)

We obtain from (1.28), (1.29) and (1.30) the estimates,

\[ \| U^{\e}_{m} \|_{L^{\infty}(0, T; H^{1}(O))} \leq C, \quad \forall m \in \mathbb{N}, \ \varepsilon > 0. \]

(1.31)
where $C$ is a constant independent of $m \in \mathbb{N}$ and $\varepsilon > 0$.

By the estimates (1.31)-(1.34), there exist a subsequence of $(U_m^\varepsilon)$, still denoted by $(U_m^\varepsilon)$, and a function $U^\varepsilon$ such that

$$U_m^\varepsilon \to U^\varepsilon \text{ weak-star in } L^\infty(0,T;H_0^1(\Omega)),$$

(1.35)

$$\frac{1}{\sqrt{\varepsilon}} M \frac{\partial}{\partial t} U_m^\varepsilon \to \frac{1}{\sqrt{\varepsilon}} M \frac{\partial}{\partial t} U^\varepsilon \text{ weak-star in } L^\infty(0,T;L^2(\Omega)).$$

(1.37)

**THE NONLINEAR TERM.**

By (1.30) and noting that $\frac{1}{p} + \frac{1}{p'} = 1$, we obtain

$$\left\| |U_m^\varepsilon|^p U_m^\varepsilon\right\|_{L^{p'}} = \int_\Omega |U_m^\varepsilon|(p+1)p' \, dx = \int_\Omega |U_m^\varepsilon|(p-1)p' \, dx = \int_\Omega |U_m^\varepsilon|^p \, dx \leq C,$$

which implies:

$$\left\| |U_m^\varepsilon|^p U_m^\varepsilon\right\|_{L^\infty(0,T;L^{p'}(\Omega))} \leq C, \quad \forall m \in \mathbb{N}, \quad \forall \varepsilon > 0.$$

(1.38)

From (1.31), (1.32) and the Aubin-Lions Theorem (see [7]) we obtain:

$$|U_m^\varepsilon|^p U_m^\varepsilon \to |U^\varepsilon|^p U^\varepsilon \text{ a.e. in } \Omega \times (0,T),$$

(1.39)

and

$$|U_m^\varepsilon|^p U_m^\varepsilon \to W \text{ weak-star in } L^\infty(0,T;L^{p'}(\Omega)).$$

(1.40)

The difficulty is to prove that $W = |U^\varepsilon|^p U^\varepsilon$. This is a consequence of the following result due to W.A. Strauss (see [15]).

**LEMMA 1.** Let $\Omega$ be a bounded open set of $\mathbb{R}^n$. Let $g_m$ and $g \in L^p(\Omega)$, $1 < p < \infty$ satisfy the following conditions:

(i) $g_m \to g$ a.e. in $\Omega$

(ii) $\|g_m\|_{L^p(\Omega)} \leq C$, $\forall m \in \mathbb{N}$.

Then:

(iii) $g_m \to g$ strongly in $L^q(\Omega)$, $1 \leq q < p$

(iv) $g_m \to g$ weakly in $L^p(\Omega)$.

Lemma 1 with $q = \frac{p+2}{p+1} = p'$; $\Omega = \Omega \times (0,T)$ and $g_m = |U_m^\varepsilon|^p U_m$, we obtain from (1.38) and (1.39) that

$$|U_m^\varepsilon|^p U_m^\varepsilon \to |U^\varepsilon|^p U^\varepsilon \text{ weak-star in } L^\infty(0,T;L^{p'}(\Omega))$$

(1.41)

and consequently weak in $L^{p'}(0,T;L^{p'}(\Omega))$.

By multiplying both sides of (1.19) by $\theta \in C^\infty_0(0,T)$, integrating from $t = 0$ to $t = T$, passing to the limit and using the convergences (1.35)-(1.37), (1.41) and noting that $\{w_p\}_{p=1}^\infty$ is a basis of $H_0^1(\Omega)$, we obtain:
\[ \int_0^T (\dot{k}_{2x}(x) \frac{\partial^2}{\partial t^2} U^\varepsilon(t), v) dt + \int_0^T (\dot{k}_{i_1}(x) \frac{\partial}{\partial t} U^\varepsilon(t), v) dt + \int_0^T a(t, U^\varepsilon(t), v) dt + \int_0^T \frac{1}{2} M \frac{\partial}{\partial t} U^\varepsilon(t), v) dt + \int_0^T (1 - \frac{\partial}{\partial t}) U^\varepsilon(t), v) dt = \int_0^T (f(t), v) dt. \]  

(1.42)

\( \forall v \in H^1_0(O), \forall \theta \in C_0^\infty(0,T). \)

Then, from (1.35)-(1.37) and from (1.42), we obtain \( U^\varepsilon \) satisfying (1.9)-(1.10) and (1.12).

Noting that
\[ L^2(O; L^2(O)) \rightarrow L^2(O; H^{-1}(O)), \]
we obtain
\[ -\frac{1}{2} M U'^\varepsilon - \dot{k}(x) U'^\varepsilon \in L^2(0,T; H^{-1}(O)). \]

The fact that \( a_{ij}(x, t) \frac{\partial}{\partial x_i} U(t) \in L^2(O) \) implies that
\[ \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( a_{ij}(x, t) \frac{\partial}{\partial x_i} U(t) \right) \in L^2(0,T; H^{-1}(O)), \]
(see [3]). Also from (1.16), (1.41) and \( \tilde{f} \in L^2(0,T; L^2(O)) \) we obtain
\[ \dot{k}_{2x}(x) \frac{\partial^2}{\partial t^2} U^\varepsilon \in L^2(0,T; H^{-1}(O)), \]
which proves (1.15).

The estimates (1.31)-(1.34) and (1.38) are independent form \( \varepsilon > 0 \), we obtain the same convergences (1.35)-(1.37) and (1.41) by changing \( U^\varepsilon \) by \( U \) and \( U^\varepsilon \) by \( W \). Therefore, we have
\[ U^\varepsilon \rightarrow W \text{ weak-star in } L^\infty(0,T; H^1_0(O)) \]  
(1.43)
\[ U_i^\varepsilon \rightarrow W_i \text{ weak in } L^2(0,T; L^2(O)) \]  
(1.44)
\[ \sqrt{k_{2x}(x)} U_i^\varepsilon \rightarrow \sqrt{k_{2x}(x)} W_i \text{ weak-star in } L^\infty(0,T; L^2(O)). \]  
(1.45)

Note that \( \sqrt{k_{2x}(x)} = \sqrt{k_{2x}(x) + \varepsilon - \varepsilon} \sqrt{k_{2x}(x)} \) strong in \( L^2(0,T; L^2(O)). \)

\[ |U^\varepsilon|^2 U^\varepsilon \rightarrow |W|^2 W \text{ weak-star in } L^\infty(0,T; L^2(O)) \]  
(1.46)

Also, we obtain the essential estimates:
\[ \int_{O \times (0,T)} M(U_i^\varepsilon) dx dt \leq Ce. \]  
(1.47)

From (1.44) we have: \( M(U_i^\varepsilon)^2 \rightarrow M(W_i)^2 \text{ weak in } L^2(0,T; L^2(O)). \)

Therefore, from (1.47) we obtain
\[ \int_{O \times (0,T)} M(W_i)^2 dx dt = 0. \]

From this and the definition of \( M \), we deduce: \( W_i = 0 \text{ a.e. in } O \times (0,T) \supset Q \). Consequently \( W(x,t) \) is constant in the variable \( t \) in \( O \times (0,T) \supset Q \). Being \( W(x,0) = \tilde{u}_0(x) \) in \( O \), we conclude that \( W(x,0) = 0 \) in \( O \setminus Q \). From this and from (H-1), we get:

\[ W(x,t) = 0 \text{ a.e. in } O \times (0,T) \supset Q. \]  
(1.48)

We conclude from (1.43) and (1.44) that \( W(t) \in H^1(O) \). Let \( u \) be the restriction of \( W \) to \( Q \).
Then from (1.48) and (H-2), we obtain that \( u \in L^\infty(0,T; H^1_0(\Omega_t)) \); which proves (1.4) in Theorem 1. Moreover, from (1.44) and (1.45), we conclude that \( u \) satisfies (1.5).

Let \( \hat{U} \) be the restriction of \( U \) to \( Q \). Then, restricting the equation of Theorem 2 to the domain \( Q \), we obtain:

\[
(k_2(x)\hat{U}'_1(t), v) + (k_1(x)\hat{U}'_2(t), v) + a(t, \hat{U}'_1(t), v) + \frac{1}{\varepsilon} (M\hat{U}'_1(t), v) + \frac{1}{\varepsilon} (|\hat{U}'_1(t)| \varepsilon\hat{U}'_1(t), v) = \langle \hat{f}(t), v \rangle,
\]

\( \forall v \in H^1_0(\Omega) \), in the sense of the \( D'(0,T) \).

By taking the limit when \( \varepsilon \to 0 \) in (1.49), and using the convergences (1.43)-(1.46) we get:

\[
\frac{d}{dt} (k_2(x)u_1(t), v) + (k_1(x)u_1(t), v) + a(t, u(t), v) + \langle \varepsilon u(t) | \varepsilon u(t), v \rangle = \langle f(t), v \rangle,
\]

in \( D'(0,T), \forall v \in H^1_0(\Omega_t) \), which proves (1.7).

The proof of (1.6) is analogous to (1.15) of the cylinder problem.

(iii) The Initial Conditions.

Let \( \sigma \in C^1([0,T]; \mathbb{R}) \) be such that \( \sigma(0) = 1 \) and \( \sigma(T) = 0 \). We have

\[
\int_0^T \left( \frac{\partial}{\partial t} U'_m(t), v \right) \sigma(t) dt = -(U'_m(0), v) - \int_0^T (U'_m(t), v) \sigma'(t) dt, \quad \forall v \in L^2(\Omega).
\]

By passing to the limit in the above equality and using the convergences (1.20), (1.35) and (1.36) we obtain:

\[
\int_0^T \left( \frac{\partial}{\partial t} U'(t), v \right) \sigma(t) dt = -(\hat{u}_0, v) - \int_0^T (U'(t), v) \sigma'(t) dt, \quad \forall v \in L^2(\Omega).
\]

Integrating by parts the last integral above, we conclude that

\[
(U'(0), v) = (\hat{u}_0, v), \forall v \in L^2(\Omega).
\]

From this it follows (1.17). The initial condition \( u(x,0) = u_0(x) \) of Theorem 1 is done analogously.

Finally, we will verify condition (1.18). Initially we verify that \( [(k_2(x) + \varepsilon)U'_1](0) \) does make sense.

Let \( U' \) be a solution of the perturbated problem. Then

\[
- \int_0^T \left( k_2(x)U'_1(t), \theta'(t) v \right) dt + \int_0^T \left( k_1(x)U'_2(t), \theta(t) v \right) dt + \int_0^T \left( A(t)U'(t), \theta(t) v \right) dt + \int_0^T \left( \frac{1}{\varepsilon} M U'_1(t), \theta(t) v \right) dt + \int_0^T \left( |U'(t)| \varepsilon U'(t), \theta(t) v \right) dt = \int_0^T \langle \hat{f}(t), \theta(t) v \rangle dt
\]

\( \forall v \in H^1_0(\Omega) \) and \( \forall \theta \in C_0^\infty(0,T) \); where \( \langle \cdot, \cdot \rangle \) is the duality between \( H^1_0(\Omega) \) and \( H^{-1}(\Omega) \). So

\[
- \int_0^T k_2(x)U'_1(t)\theta'(t) dt + \int_0^T k_1(x)U'_2(t)\theta(t) dt + \int_0^T A(t)U'(t)\theta(t) dt + \int_0^T \frac{1}{\varepsilon} M U'_1(t)\theta(t) dt + \int_0^T |U'(t)| \varepsilon U'(t)\theta(t) dt = \int_0^T \langle \hat{f}(t), \theta(t) v \rangle dt
\]

\( \forall v \in H^1_0(\Omega) \) and \( \forall \theta \in C_0^\infty(0,T) \).
Therefore, we have
\[
< -\hat{k}_2(x)U_\gamma'(t),\theta'(t) > + < \hat{k}_1(x)\mathbf{u}'_\gamma(t),\theta(t) > + < A(t)|\mathbf{u}'_\gamma(t)|,\theta(t) > + \\
< \frac{1}{2} MU'_\gamma(t),\theta(t) > + < |U'_\gamma(t)|^p U'_\gamma(t),\theta(t) > = < \hat{f}(t),\theta(t) >.
\]
\(\forall \theta \in C^\infty_0(0,T)\); where, here \(< \cdot, \cdot >\) denotes the vectorial distribution of \((0,T)\) in \(H^{-1}(0)\) evaluated in scalar test application of \((0,T)\). Being \(\hat{k}_2 \in L^\infty(O \times (0,T))\) and \(U'_\gamma \in L^2(0,T;L^2(O))\), we have \(-\hat{k}_2 U'_\gamma \in L^2(0,T;L^2(O))\).

So \(-\hat{k}_2 U'_\gamma\) defines a vectorial distribution of \((0,T)\) in \(L^2(O)\), whose derivative is:
\[
< -\hat{k}_2 U'_\gamma,\theta' > = < (\hat{k}_2 U'_\gamma)_\gamma,\theta >, \ \forall \theta \in C^\infty_0(0,T).
\]

Therefore,
\[
< (\hat{k}_2 U'_\gamma)_\gamma,\theta > + < \hat{k}_1 U'_\gamma,\theta > + < A(t)U'_\gamma,\theta > + \\
< \frac{1}{2} MU'_\gamma,\theta > + < |U'_\gamma|^p U'_\gamma,\theta > = < \hat{f},\theta >, \forall \theta \in C^\infty_0(0,T).
\]

Or,
\[
(\hat{k}_2 U'_\gamma)_\gamma + \hat{k}_1 U'_\gamma + A(t)U'_\gamma + \frac{1}{2} MU'_\gamma + |U'_\gamma|^p U'_\gamma = \hat{f},
\]

in \(L^2(0,T;H^{-1}(0))\). As \(\hat{f},\hat{k}_1 U'_\gamma,\frac{1}{2} MU'_\gamma, |U'_\gamma|^p U'_\gamma \in L^2(0,T;L^2(O))\) and \(A(t)U'_\gamma \in L^2(0,T;H^{-1}(0))\), we obtain, from the last equality above that: \((\hat{k}_2 U'_\gamma)_\gamma \in L^2(0,T;H^{-1}(0)) \rightarrow L^p(0,T;H^{-1}(0))\), which proves (1.15). It is easy to see that \(\hat{k}_2 U'_\gamma \in C^0(0,T;H^{-1}(0))\). Therefore, \([\hat{k}_2 U'_\gamma](0)\) makes sense. Let now \(\theta \in C^1([0,T],\mathbb{R})\) be such that \(\theta(0) = 1\) and \(\theta(T) = 0\). Then,
\[
\int^T_0 (\hat{k}_2 \frac{\partial^2}{\partial t^2} U'_\gamma(t),v)\theta(t)dt = -\left(\hat{k}_2 \frac{\partial}{\partial t} U'_\gamma(0),v\right) \\
- \int^T_0 (\hat{k}_2 \frac{\partial}{\partial t} U'_\gamma(t),v)\theta'(t)dt, \forall v \in V_m.
\]

From this and taking \(v = w_j\) in the approximate equation, we obtain:
\[
-\left(\hat{k}_2 \frac{\partial}{\partial t} U'_\gamma(0),v\right) - \int^T_0 (\hat{k}_2 \frac{\partial}{\partial t} U'_\gamma(t),v)\theta'(t)dt + \int^T_0 \left(\frac{1}{2} M \frac{\partial}{\partial t} U'_\gamma(t),v\right)\theta'(t)dt + \\
\int^T_0 a(t,U'_\gamma(t),v)\theta(t)dt + \int^T_0 \left(\frac{1}{2} MU'_\gamma(t),v\right)\theta(t)dt + \int^T_0 (|U'_\gamma(t)|^p U'_\gamma(t),v)\theta(t)dt = \\
\int^T_0 (\hat{f}(t),v)\theta(t)dt, \ \forall v \in V_m.
\]

By passing to the limit in the above equality and using the convergences (1.21), (1.35)-(1.37) and (1.41) we obtain:
\[
-\left(\sqrt{\hat{k}_2} \mathbf{u}',v\right) - \int^T_0 (\hat{k}_2 \mathbf{u}'_\gamma(t),v)\theta'(t)dt + \int^T_0 (\hat{k}_1(t)\mathbf{u}'_\gamma(t),v)\theta(t)dt + \\
\int^T_0 a(t,U'_\gamma(t),v)\theta(t)dt + \int^T_0 \left(\frac{1}{2} MU'_\gamma(t),v\right)\theta(t)dt + \\
\int^T_0 (|U'_\gamma(t)|^p U'_\gamma(t),v)\theta(t)dt = \int^T_0 (\hat{f}(t),v)\theta(t)dt,
\]
As \( - \int_{\Omega} (\hat{k}_2 \theta'(t), v) \theta(t) \, dt = \langle \hat{k}_2 U'_\theta(t), v \rangle > \theta(t) \theta(v) \forall v \in V_m \) and \( \theta \in C^1([0,T];\mathbb{R}) \) such that \( \theta(0) = 1 \) and \( \theta(T) = 0 \), we have, using the fact that \( U' \) is solution of the perturbed equation, that:

\[
- \langle \sqrt{\hat{k}_2(x)} \bar{u}_1, v \rangle + \langle \hat{k}_2(x) U'_\theta(0), v \rangle = 0, \forall v \in V_m.
\]

Or,

\[
\langle \hat{k}_2(x) U'_\theta(0) - \sqrt{\hat{k}_2(x)} \bar{u}_1, v \rangle = 0,
\]

\( \forall v \in H^1_0(\Omega) \). This proves (1.18) and, therefore, the proof of Theorem 2 is complete.

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