ABSTRACT. Let $X$ be a metric space and let $CB(X)$ denote the closed bounded subsets of $X$ with the Hausdorff metric. Given a complete subspace $Y$ of $CB(X)$, two fixed point theorems, analogues of results in [1], are proved, and examples are given to suggest their applicability in practice.

KEY WORDS AND PHRASES. Fixed Point Theorems

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Let $X$ be a metric space with metric $d$ and let $Y$ be a complete subspace of the space $CB(X)$ of all closed and bounded subsets of $X$, with the Hausdorff metric $p$:

$$p(A, B) = \max \{\sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B)\}. \quad (1)$$

In Hicks [1], fixed point theorems for set-valued maps $T : X \rightarrow CB(X)$ were proved; and illustrated with examples. We show that similar results for maps $T : Y \rightarrow X$ can be obtained, using essentially the same techniques as in Hicks [1].

THEOREM 1. Let $T : Y \rightarrow X$ be continuous. Then there is an $A \in Y$ such that $T(A) \subseteq A$ iff there exists a sequence $\{A_n\}_{n=0}^{\infty}$ in $Y$ with $T(A_n) \subseteq A_{n+1}$ (or $T(A_{n+1}) \subseteq A_n$) and

$$\sum_{n=0}^{\infty} p(A_n, A_{n+1}) < \infty. \quad (2)$$

In this case, $A_n \rightarrow A$ as $n \rightarrow \infty$. (In fact, we may let $A_{n+1} = A_n \cup \{T(A_n)\}$, for each $n$, for the case $T(A_n) \subseteq A_{n+1}$.)

PROOF. If $T(A) \subseteq A$, then we are done. Conversely, if the given conditions are met, then $\{A_n\}_{n=0}^{\infty}$ is Cauchy, so let $A \in Y$ be its limit. Thus $T(A_n) \rightarrow T(A)$. If $y \in A$, then

$$d(y, T(A)) \leq d(y, T(A_n)) + d(T(A_n), T(A)), \quad (3)$$

so

$$d(A, T(A)) \leq d(A, T(A_n)) + d(T(A_n), T(A)). \quad (4)$$
Since $d(T(A_n), T(A)) \to 0$ and we have $d(A, T(A_n)) \leq \rho(A, A_{n+1}) \to 0$, it follows that $T(A) \in A$.

**EXAMPLES**

1. Let $X = \mathbb{R}$, with the usual metric. Define $T : CB(\mathbb{R}) \to \mathbb{R}$ by
   \[ T(A) = \alpha \sup(A) + (1 - \alpha) \inf(A), \]
   where $\alpha \in [0,1]$. Then $T$ is continuous. If $A \in CB(\mathbb{R})$, then
   \[ T(A \cup \{T(A)\}) = T(A) \in A \cup \{T(A)\}. \]

2. Let $X = \mathbb{R}$ as in 1, and let $r : [0,\infty) \to (0,\infty)$ be such that $r - 1_\mathbb{R}$, where $1_\mathbb{R}$ is the identity on $\mathbb{R}$. Define $T : CB(\mathbb{R}) \to \mathbb{R}$ by
   \[ T(A) = r(\sup(A)) + (1 - \alpha) r(\inf(A)), \]
   where $\alpha \in (0,1)$. Assuming $r$ is continuous, so is $T$. Let $A_0 \in CB(\mathbb{R})$, and for $n \in \mathbb{N}$, let
   \[ A_{n+1} = A_n \cup \left[ \inf \{T(A_n)\}, \sup \{T(A_k)\} \right]. \]
   Theorem 1 yields $A \in CB(\mathbb{R})$ with $T(A) \in A$ if
   \[ \sum_{n=1}^\infty \max \left\{ d \left( \inf \{T(A_n)\}, A_n \right), d \left( \sup \{T(A_k)\}, A_n \right) \right\} < \infty. \]

**DEFINITION.** Let $(X, d)$ be a metric space and let $Y$ be a subspace of $(CB(X), \rho)$. Let $T : Y \to X$. Then $T$ is nice if for each $A \in Y$ and each $x \in A$ with $d(x, T(A)) = d(A, T(A))$, there exists a set $B \in Y$ with $T(B) = x$.

**EXAMPLES**

3. Let $X = \mathbb{R}^2$, $T : CB(\mathbb{R}^2) \to \mathbb{R}^2$ defined by
   \[ T(A) = (\inf proj_1(A), \sup proj_1(A)). \]
   Let $a > b$ and $A = [0,a] \times [0,b]$. Then $T(A) = (0,a)$, and $(0,b)$ is the only point of $A$ whose distance from $(0,a)$ equals $d(A, T(A))$. Let $B = [0,b]^2$. Then $T(B) = (0,b)$.

4. Let $X = \mathbb{R}^2$, and for $A \in CB(\mathbb{R}^2)$, let $T(A)$ be the center of the circle which circumscribes $A$. Let $r = d(A, T(A))$, and let $x \in A$ with $d(x, T(A)) = r$. Let $B = A \cap B \left( x, \frac{\text{diam}(A)}{2} \right)$. Then $T(B) = x$.

**THEOREM 2.** Let $(X, d)$ be a metric space and let $Y$ be a complete subspace of $(CB(X), \rho)$, each member of which is compact. Let $T : Y \to X$ be continuous. Assume that $K : [0,\infty) \to [0,\infty)$ is non-decreasing, $K(0) = 0$, and
   \[ \rho(A, B) \leq K(d(T(A), T(B))) \]
   for $A, B \in Y$. If $T$ is nice, then there is $A \in Y$ such that $T(A) \in A$ iff there exists $A_0 \in Y$ for which
\[ \sum_{n=1}^{\infty} K^n(d(A_0, T(A_0))) < \infty \quad \text{(\star)} \]

In this case, we can choose \( \{A_n\}_{n=1}^{\infty} \) such that \( T(A_{n+1}) \in A_n \) and \( A_n \to A \).

**PROOF.** If \( T(A) \in A \), then we are done. If \( A_0 \in Y \) satisfies (\star), let \( x_1 \in A_0 \) with \( d(x_1, T(A_0)) = d(A_0, T(A_0)) \). Since \( T \) is nice, let \( A_1 \in Y \) with \( T(A_1) = x_1 \).

Next, let \( x_2 \in A_1 \) with \( d(x_2, T(A_1)) = d(A_1, T(A_1)) \), and then let \( A_2 \in Y \) with \( T(A_2) = x_2 \). Then

\[
\begin{align*}
  d(T(A_1), T(A_2)) &= d(T(A_1), x_2) \\
  &= d(T(A_1), A_1) = d(x_1, A_1) \\
  &\leq \rho(A_0, A_1) \leq K(d(T(A_0), T(A_1))).
\end{align*}
\]

so that

\[
K(d(T(A_1), T(A_2))] \leq K^2[d(T(A_0), T(A_1))] = K^2[d(T(A_0), x_1)] = K^2[d(T(A_0), A_0)].
\]

Now, suppose we have \( x_n \in A_{n-1} \) and \( A_n \in Y \) with \( d(x_n, T(A_{n-1})) = d(A_{n-1}, T(A_{n-1})) \) and \( T(A_n) = x_n \). Let \( x_{n+1} \in A_n \) with \( d(x_{n+1}, T(A_n)) = d(A_n, T(A_n)) \) and let \( A_{n+1} \in Y \) with \( T(A_{n+1}) = x_{n+1} \). Then

\[
\begin{align*}
  d(T(A_n), T(A_{n+1})) &= d(T(A_n), x_{n+2}) \\
  &= d(T(A_n), A_n) = d(x_n, A_n) \\
  &\leq \rho(A_{n-1}, A_n) \leq K(d(T(A_{n+1}), T(A_{n+1}))).
\end{align*}
\]

so that

\[
K[d(T(A_n), T(A_{n+1}))] \leq K^2[d(T(A_{n-1}), T(A_n))] = K^2[d(T(A_{n-1}), T(A_{n-1}))] \\
= K^2[d(T(A_{n-2}), T(A_{n-1}))] \leq K^2[d(T(A_{n-2}), T(A_{n-2}))] = K^3[d(T(A_{n-2}), T(A_{n-2}))] \\
\leq \cdots \leq K^n[d(T(A_0), A_0)].
\]

Thus, since

\[
\rho(A_n, A_{n+1}) \leq K(d(T(A_n), T(A_{n+1}))),
\]

it follows from (\star) that

\[
\sum_{n=0}^{\infty} \rho(A_n, A_{n+1}) < \infty,
\]

and then by Theorem 1, \( A_n \to A \) and \( T(A) \in A \).
Note that the conditions of theorem 2 force $T$ to be a bijection. In both of these theorems, we have used completeness of the given subspace $Y$ of $CB(X)$ instead of completeness of $X$. In fact, in theorem 2, since $T$ is a bijection, we may trade completeness of $Y$ back for completeness of $X$ and use the second theorem of Hicks [1].

**THEOREM 3.** If $(X,d)$ is a complete metric space and $Y$ is any subspace of $(CB(X),\rho)$, each member of which is compact, then for any homeomorphism $T : Y \to X$ such that

$$\rho(A,B) \leq K(d(T(A),T(B))),$$

where $K : [0,\infty) \to [0,\infty)$ is nondecreasing, with $K(0) = 0$, there is $A \in Y$ such that $T(A) \in A$ iff there exists $A_0 \in Y$ for which (*) holds.

**PROOF.** If $A_0 \in Y$ satisfies (*), let $x_0 = T(A_0)$. Apply theorem 2 of Hicks [1] to $T^{-1} : X \to Y$ to obtain a $p \in X$ such that $p \in T^{-1}(A_0)$, $A = T^{-1}(p)$. Then $T(A) = p$ is in $A$, so we are done. ■

**REFERENCES**