ON CERTAIN SEQUENCE SPACES II

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ABSTRACT In this paper we define the space $c_0(L) = \{x = (x_k) / x_{k-1} \to 0 \text{ (k} \to \infty) , x_0 = 0 , x_k \in C \}$ and compute its duals (Continuous dual, $\beta$-dual and $N$-dual) The aim of this paper is to give some results about matrix mapping of $c_0(L)$ into other sequence spaces including the convergent sequences, null sequences and bounded sequences

KEY WORDS AND PHRASES: Sequence spaces, matrix maps, $\Delta$-norm, $\beta$-dual, Null-dual

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1. Introduction

Let $l_\infty$, $c$ and $c_0$ be the linear spaces of complex bounded, convergent and null sequences $x = (x_k)$ respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in \mathbb{N} = \{1,2,\ldots\}$ the positive integers. On the other hand we defined $l_0(L) = \{x = (x_k) / \Delta x \in l_0 \}$, $c(L) = \{x = (x_k) / \Delta x \in c \}$ and $c_0(L) = \{x = (x_k) / \Delta x \in c_0 \}$ where $\Delta x = (x_k - x_{k-1})$, $x_0 = 0$ [2]. (Throughout this paper it is assumed that $x_0 = 0$)

$c_0(L)$, $c(L)$ and $l_0(L)$ are Banach Spaces with the norm

$$\|x\|_L = \sup_k |x_k| - x_{k-1}|$$

$c_0$, $c$, $l_\infty$ and $M_0 = l_\infty \cap c_0(L)$ are Banach with the norm $\|\|_\infty$ but they aren't Banach with the norm $\|\|_L$.

If we say $sx = ( \sum_{k=1}^n x_k )$ then we have $m_s = (x = (x_k) / sx \in l_\infty)$, $c_s = (x = (x_k) / sx \in c)$ and $(c_0)_s = (x = (x_k) / sx \in c_0)$ [4]. $l_\infty$, $c$ and $c_0$ are isometrically isomorphic to $m_s$, $c_s$ and $(c_0)_s$, respectively with their natural norms.

For instance $f: l_\infty \to m_s$, $f(x) = \Delta x$ and $f^{-1}: m_s \to l_\infty$, $f^{-1}(x) = sx$ are isometric isomorphisms. Similarly $l_\infty(L)$, $c(L)$ and $c_0(L)$ are isometrically isomorphic to $l_\infty$, $c$ and $c_0$ respectively. Obviously

$$f: (c_0(L), \|\|_\infty) \to (c_0, \|\|_\infty), f(x) = \Delta x$$

and

$$f^{-1}: (c_0, \|\|_\infty) \to (c_0(L), \|\|_\infty), f(x) = sx$$

are isometric isomorphisms.

We have investigated matrix maps and related questions connected with $l_\infty(L)$ and $c(L)$ in [2]. We know that $c_0(L)$ and $c(L)$ have Schauder basis but $l_\infty$ has no basis with the norm $\|\|_\infty$. Write $e_k = (0,0,\ldots,0,1,0,\ldots)$. Then $(e_k)$ is a basis for $c_0(L)$ and $(e_{k-1})$ $(e_0 = (1,1,1,\ldots))$ is a basis for $c(L)$ with $\|\|_\infty$ and $\|\|_L$. On the other hand $(E_k = (0,0,1,1,\ldots))$ is a basis for $M_0$ and $c_0(L)$ with the norm $\|\|_L$. So $c_0(L)$ is a separable Banach Space.

We know that the continuous dual of $c_0(L)$ and $c(L)$ is $l_1 = \{x = (x_k) / \sum_{k=1}^\infty |x_k| < \infty, x_k \in C \}$ [3] (Page 110) ($C$ the set of complex numbers). Thus $l_1$, is continuous dual of $c_0(L)$ by (1.1) Moreover, we can prove that
\[ \overline{c_0} = c_0(\Delta) \]

with the norm \( \| \cdot \|_\Delta \), where the bar denotes closure. For this, let \( x \in c_0(\Delta) \) and \( \varepsilon > 0 \) be any number. Then there exists one and only one \( y = (y_k) \in c_0 \) such that \( x_k = \sum_{i=1}^{k} y_i \) (1.1) and a corresponding index \( M = M(\varepsilon) \in \mathbb{N} \) such that \( |y_k| < \varepsilon/2 \) for all \( k \geq M \). Now we take

\[ x_k, \quad 1 \leq k \leq M \]

\[ z_k = x_k, \quad k > M \]

thus \( z = (z_k) \in c_0(\Delta) \) belongs to the open ball \( B(x, \varepsilon) \) which is in \( (c_0(\Delta), \| \cdot \|_\Delta) \)

**2. \( \beta \)-dual, N-dual and Matrix Maps**

If \( X \) is a sequence space, we define

\( X^\beta = \{ a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x \in X \} \)

\( X^N = \{ a = (a_k) : \lim_{k \to \infty} a_k x_k = 0, \text{ for each } x \in X \} \). \( X^\beta \) is called the \( \beta \)-dual (or generalized Köthe-Toeplitz dual) \[1\] and we will say that \( X^N \) is the \( N \)-dual space of \( X \). We have that if \( X \subset Y \), then \( Y^N \subset X^N \) and \( X^N \subset Y^N \). The \( N \)-dual has similar properties with the \( \beta \)-dual. For instance if \( X \subset Y \) then \( Y^N \subset X^N \) and \( X^N \subset Y^N \).

Obviously \( c_0^N = l_1^N = \infty = M_0^N = N = c_N = c_0^N \).

\( c_N(\Delta) = l_1^N(\Delta) = \{ a = (a_k) / (k a_k) \in c_0 \} \). Let \( (X, Y) \) denote the set of all infinite matrices \( A = (a_{nk}) \) which map \( X \) into \( Y \).

**Lemma 1.** Let \( (a_k) \in l_1 \) and if \( \lim_{k \to \infty} |a_k x_k| = L \) exists for an \( x \in c_0(\Delta) \), then \( L = 0 \).

**Proof.** It is trivial if \( x = (x_k) \) is bounded. Suppose that \( x \in c_0(\Delta) \) is unbounded. If \( (x_k) \) is bounded then \( \lim_{n \to \infty} |a_k x_k| = 0 \) implies \( L = 0 \). So we can take \( x_k = 0 \) for all \( n \in \mathbb{N} \).

Now let \( \varepsilon = 1/2 > 0 \), then there exists an \( M_1 = M_1(\varepsilon) \in \mathbb{N} \) such that \( \frac{1}{2} < |a_k x_k| < \frac{3L}{2} \) for all \( k \geq M_1 \). Thus we get

\[ \frac{1}{2} |a_k| > L \]

for all \( k \geq M_1 \) and

\[ \sum_{k=1}^{\infty} \frac{1}{|a_k|} < \infty \]  \( (2.1) \)

We have that \( x_k \to 0 \) \( (k \to \infty) \) \[2\]. Let \( \varepsilon = 1 \), then we have \( \frac{1}{|a_k|} < 1 \) and \( \frac{1}{|a_k|} > \frac{1}{k} \) for all \( k \geq M_1 \) \( (1) \in \mathbb{N} \). If we take max \( \{ M_1, M_2 \} = M \) then \( \sum_{k=1}^{\infty} \frac{1}{|x_k|} = \sum_{k=M}^{\infty} \frac{1}{|x_k|} \). This contradicts with \( (2.1) \). So \( L \) must be zero.

**Lemma 2.** \( c_0^N(\Delta) = \{ a = (a_k) / (k a_k) \in c_0 \} = E \).

**Proof.** Suppose that \( a = (a_k) \in E \). Since \( \lim_{k \to \infty} \frac{x_k}{k} = 0 \) for all \( x = (x_k) \in c_0(\Delta) \) \[2\], then we get \( \lim a_k x_k = \lim k a_k \frac{x_k}{k} = 0 \). This implies that \( a \in c_0^N(\Delta) \).

Now let \( a \in c_0^N(\Delta) \). Then \( \lim_{k \to \infty} a_k x_k = 0 \) for all \( x \in c_0(\Delta) \), then there exists one and only one \( y = (y_k) \in c_0^N \),
such that \( x_n = \sum_{k=1}^{n} y_k \) (1.1)

\[
\lim_{n} a_n x_n = \lim_{n} \sum_{k=1}^{n} a_n y_k = 0 \quad \text{for all } y=(y_k) \in c_0. \quad \text{If we take}
\]

\[
a_{nk} = \begin{cases} 
    a_n, & 1 \leq k \leq n \\
    0, & k > n
\end{cases}
\]

we get \( \lim_{n} \sum_{k=1}^{n} a_{nk} y_k = 0, \) for all \( x \in c_0. \) Then \( A = (a_{nk}) \in (c_0, c_0) \) and we have

\[
\sup_{n} \sum_{k=1}^{\infty} |a_{nk}| = \sup_{n} \sum_{k=1}^{n} |a_{nk}| = \sup_{n} n |a_{nk}| < \infty \quad [4] \quad \text{This completes the proof.}
\]

For the next results we introduce the sequence \((R_k)\) [resp. matrix \(R\)] given by \( R_k = \sum_{i=k}^{\infty} a_i \) [resp. matrix

\[
R = (R_{nk}) = (\sum_{i=k}^{\infty} a_{ni})
\]

**LEMMA 3.** \( c_0^\beta(\Delta) = \{a=(a_k) \in l_1 \mid (R_k) \in l_1 \cap c_0^\beta(\Delta) \} = D \)

**Proof.** Suppose that \( a \in D. \) If \( x \in c_0(\Delta) \) then we use Abel's summation formula to get

\[
\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} \left( \sum_{i=1}^{k} a_i \right) x_k + \sum_{k=1}^{n} a_k x_{k+1}
\]

\[
= \sum_{k=1}^{n} (R_1 \cdot R_{k+1}) x_k + (R_1 \cdot R_{n+1}) x_{n+1}
\]

\[
= \sum_{k=1}^{n+1} R_k (x_k - x_{k-1}) - R_{n+1} x_{n+1} \quad (2.2)
\]

This implies that \( \sum_{k=1}^{\infty} a_k x_k \) is convergent, then \( a \in c_0^\beta(\Delta). \)

If \( a \in c_0^\beta(\Delta) \) then \( \sum_{k=1}^{\infty} a_k x_k \) is convergent for all \( x \in c_0(\Delta) \) Obviously \( a \in l_1 \) if \( x \in c_0(\Delta), \) then there exists \( y=(y_k) \in c_0 \) such that \( x_k = \sum_{i=1}^{k} y_i \) (1.1)

Then

\[
\sum_{k=1}^{n} R_k y_k = \sum_{k=1}^{\infty} \left( \sum_{i=1}^{k} y_i \right) a_k + R_{n+1} \sum_{k=1}^{n} y_k \quad \text{with Abel summation formula} \quad \text{Thus we have}
\]

\[
\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} (R_k \cdot R_{n+1}) y_k = \sum_{k=1}^{n} \left( \sum_{i=k}^{\infty} a_i \right) y_k \quad (2.3)
\]

If we take

\[
a_{nk} = \begin{cases} 
    \sum_{i=k}^{n} a_i, & 1 \leq k \leq n \\
    0, & k > n
\end{cases}
\]

then \( A = (a_{nk}) \in (c_0, c) \) since \( \lim_{n} \sum_{k=1}^{\infty} a_{nk} y_k = \lim_{n} \sum_{k=1}^{n} a_{nk} y_k \) exists for all \( y \in c_0 \) (2.3). This implies that
Sup\(\sum_{k=1}^{\infty} l_a_{nk}\) = Sup\(\sum_{k=1}^{\infty} l a_{i,k}\) [4]. Thus we get \(\sum_{k=1}^{\infty} l R_{i,k}\) \(\in\infty\). Furthermore (2.2) implies that \(\lim_{n+1} x_{n+1}^k\) exists for each \(x \in c_0(\Delta)\) then we get \(R_{n}^k \in c_0(\Delta)\) by lemma 1. This completes the proof.

**THEOREM 1.** \(A=(a_{nk}) \in (c_0(\Delta), c) \iff T_1. (R_{nk}) \in c_0^N(\Delta)\), for each \(n \in \mathbb{N}\)

**T_2.** \(R = (R_{nk}) \in (c_0, c)\)

**Proof.** If \(a \in (c_0(\Delta), c)\) then the series \(A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k\) are convergent for each \(n \in \mathbb{N}\) and for all \(x \in c_0(\Delta)\), this implies that \(\sup_{n} \sum_{k=1}^{\infty} l a_{nk} \) \(\in\infty\) and \(\lim_{n} \sum_{k=p}^{\infty} a_{nk} = a_p\) exists for each \(p \in \mathbb{N}\) [3] (page 166). From lemma 3 we have \(\sum_{k=1}^{\infty} l R_{nk} \) \(\in\infty\), \(\lim_{n} R_{nk} x_k = 0\) for each \(n \in \mathbb{N}\) and for all \(x \in (c_0(\Delta))\). This proves \(T_1\). If we write again (2.2) we get

\[
\sum_{k=1}^{m} a_{nk} x_k = \sum_{k=1}^{m+1} R_{nk} (x_k \cdot x_{k-1}) R_{n} x_{m+1}^k
\]

and

\[
A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} R_{nk} (x_k \cdot x_{k-1})
\]

This shows that \(R \in (c_0, c)\). If we use again lemma 3 and (2.5) we get the sufficiency of \(T_1\) and \(T_2\).

Similarly we can prove that

i) \(A \in (c_0(\Delta), c_0) \iff T_1\) and \(R \in (c_0, c_0)\)

ii) \(A \in (c_0(\Delta), l_\infty) \iff T_1\) and \(R \in (l_\infty, l_\infty)\)

iii) \(A \in (c_0(\Delta), M_0) \iff T_1\) and \(R \in (l_\infty, l_\infty)\) and

\[
B=(b_{nk})=(a_{nk} - a_{n-1,k+1}) \in (c_0, c_0)\]

iv) \(A \in (c_0(\Delta), c_0(\Delta)) \iff (a_{nk}) \in c_0(\Delta),\) for each \(n \in \mathbb{N}\) and \(C=(c_{nk})=(a_{nk} - a_{n-1,k}) \in (c_0(\Delta), c_0)\) (\(a_{nk}=0\))

Open questions

1) Matrix maps for \(M_0\).

2) \(M_0\) has a Schauder basis with \(\|\|_{\Delta}\). It is \((E_k)\) (we can write \(x= \sum_{k=1}^{\infty} (x_k \cdot x_{k-1}) E_k\), each \(x \in M_0\))

Then \((M_0, \|\|_{\Delta})\) is separable.

Is \(M_0\) separable or have a Schauder basis with \(\|\|\)?

3) It is obvious that \(c_0 \subseteq c \subseteq M_0 \subseteq l_\infty\) and inclusions are strict. In this order, is there a separable space \(E\) which is \(c \subset c \subseteq M_0 \subseteq l_\infty\) with the norm \(\|\|\)? If not, is \(c\) an upper bound according to separability?

**REFERENCES**


