A CHARACTERIZATION OF THE ROGERS q-HERMITE POLYNOMIALS

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ABSTRACT. In this paper we characterize the Rogers q-Hermite polynomials as the only orthogonal polynomial set which is also $D_q$-Appell where $D_q$ is the Askey-Wilson finite difference operator.

KEY WORDS AND PHRASES. Orthogonal polynomials, generating functions, Askey-Wilson operator

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1. INTRODUCTION

Appell polynomials sets $\{P_n(x)\}$ are generated by the relation

$$A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x) t^n,$$

where $A(t)$ is a formal power series in $t$ with $A(0) = 1$. This definition implies the equivalent property that

$$D P_n(x) = P_{n-1}(x), \quad D = d/dx,$$

Examples of such polynomial sets are

$$\left\{ \frac{x^n}{n!} \right\}, \left\{ \frac{B_n(x)}{n!} \right\}, \left\{ \frac{H_n(x)}{2^n n!} \right\}$$

where $B_n(x)$ is the $n$th Bernoulli polynomial and $H_n(x)$ is the $n$th Hermite polynomials generated by

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$

By an orthogonal polynomial set (OPS) we shall mean those polynomial sets which satisfy a three term recurrence relation of the form

$$P_{n+1}(x) = (A_n x + B_n) P_n(x) - C_n P_{n-1}(x), \quad (n = 0, 1, 2, \cdots)$$

with $P_0(x) = 1$, $P_{-1}(x) = 0$, and $A_n A_{n-1} C_n > 0$.

By Favard's theorem [7] this is equivalent to the existence of a positive measure $d\alpha(x)$ such that

$$\int_{-\infty}^{\infty} P_n(x) P_m(x) \, d\alpha(x) = K_n \delta_{nm}.$$
As we see from the examples (1.3) some Appell polynomials are orthogonal and some are not. This prompted Angelesco [3] to prove that the only orthogonal polynomial sets which are also Appell is the Hermite polynomial set. This theorem was rediscovered by several authors later on (see, e.g., [10]).

There were several extensions and/or analogs of Appell polynomials that were introduced later. Some are based on changing the operator $D$ in (1.2) into another differentiation-like operator or by replacing the generating relation (1.1) by a more general one. In most of these cases theorems like Angelesco’s were given. For example Carlitz [6] proved that the Charlier polynomials are the only OPS which satisfy the difference relation

$$\Delta P_n(x) = P_{n-1}(x), \quad (\Delta f(x) = f(x+1) - f(x)).$$

(1.7)

See [1] for many other references.

A new and very interesting analog of Appell polynomials were introduced recently, as a byproduct of other considerations, by Ismail and Zhang [9]. In discussing the Askey-Wilson operator they defined a new \(q\)-analog of the exponential function \(e^x\). This we describe in the next section.

### 2. NOTATIONS AND DEFINITIONS

The Askey-Wilson operator is defined by

$$D_qf(x) = \frac{\delta_qf(x)}{\delta_qx},$$

(2.1)

where \(x = \cos \theta\) and

$$\delta_qg(e^{i\theta}) = g(q^{1/2}e^{i\theta}) - g(q^{-1/2}e^{i\theta}).$$

(2.2)

We further assume that \(-1 < q < 1\) and use the notation

$$(a; q)_0 = 1, \quad (a; q)_n = (1 - a)(1 - qa) \cdots (1 - aq^{n-1}), \quad (n = 1, 2, \ldots)$$

(2.3)

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

(2.4)

There are two \(q\)-analogs of the exponential function \(e^x\) given by the infinite products

$$e_q(x) = \frac{1}{(x; q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q; q)_k},$$

(2.5)

and

$$\frac{1}{e_q(x)} = (x; q)_\infty = \sum_{k=0}^{\infty} (-1)^k \frac{q^{1/2}(k-1)}{(q; q)_k} x^k.$$

(2.6)

We shall also use the function

$$\Psi_n(x) = i^n(q^{(1-n)/2}e^{i\theta}; q)_n(q^{(1-n)/2}e^{-i\theta}; q)_n,$$

(2.7)

so that

$$\Psi_{2n}(x) = \prod_{k=0}^{n-1} \left[ 4x^2 + (1 - q^{2n-1-2k})(1 - q^{-1-2n+2k}) \right]$$

$$\Psi_{2n+1}(x) = 2x \prod_{k=0}^{n-1} \left[ 4x^2 - (1 - q^{2n-2k})(1 - q^{-2n+2k}) \right]$$

$$4x^2\Psi_n(x) = \Psi_{n+2}(x) + (1 - q^n)(1 - q^{-n-1})\Psi_n(x)$$

(2.8)
Thus
\[
\mathcal{D}_q \Psi_n(x) = 2q^{(1-n)/2} \frac{1-q^n}{1-q} \Psi_{n-1}(x).
\] (2.9)
and
\[
\mathcal{D}_q [x \Psi_n(x)] = \frac{q^{1+n}/2 - q^{-(n+1)/2}}{q^{1/2} - q^{-1/2}} 2x \Psi_{n-1}(x).
\] (2.10)
Iterating (2.9) we get
\[
\mathcal{D}_q^k \Psi_n(x) = 2^k q^{\frac{k(k+1)}{2} - \frac{k}{2} n} \frac{(q; q)_n}{(q; q)_{n-k}(1-q)^k} \Psi_{n-k}(x).
\] (2.11)
The Ismail-Zhang q-analog of the exponential function \cite{9} is
\[
\mathcal{E}(x) = \sum_{n=0}^\infty \frac{q^{n(n-1)/4}(1-q)^n}{2^n(q; q)_n} \Psi_n(x) t^n.
\] (2.12)
It follows from (2.12) and (2.9) that
\[
\mathcal{D}_q \mathcal{E}(x) = t \mathcal{E}(x).
\] (2.13)
This suggested to Ismail and Zhang to define the $\mathcal{D}_q$-Appell polynomials as those, in analogy with (1.1), defined by
\[
A(t) \mathcal{E}(x) = \sum_{n=0}^\infty P_n(x) t^n,
\] (2.14)
so that
\[
\mathcal{D}_q P_n(x) = P_{n-1}(x).
\] (2.15)
An example of such a set is the Rogers q-Hermite polynomials, \{\(H_n(x|q)\)}, (see \cite{2, 4, 8}).
\[
\prod_{n=0}^\infty \left(1 - 2xtq^n + t^2q^{2n}\right)^{-1} = \sum_{n=0}^\infty H_n(x|q) \frac{t^n}{(q; q)_n}.
\] (3.1)
They satisfy the three term recurrence relation
\[
H_{n+1}(x|q) = 2x H_n(x|q) - (1 - q^n) H_{n-1}(x|q), \quad n = 0, 1, 2, 3, ...
\] (2.17)
with $H_0(x|q) = 1$, $H_{-1}(x|q) = 0$.

3. THE MAIN RESULT

We now state our main result:

\textbf{Theorem 1.} The orthogonal polynomial sets which are also $\mathcal{D}_q$-Appell, i.e., satisfy (2.15) or (2.14) is the set of the Rogers q-Hermite polynomials.

\textbf{Proof} Let \{\(Q_n(x)\}\} be a polynomial set which is both orthogonal and $\mathcal{D}_q$-Appell. That is, \{\(Q_n(x)\}\} satisfy (2.14) and (1.5).

We next note that (2.16) implies that
\[
h_n(x|q) = \frac{(1-q)^n q^{n(n-1)/4}}{2^n(q; q)_n} H_n(x|q)
\] (3.1)
satisfy
\[
\mathcal{D}_q h_n(x|q) = h_{n-1}(x|q),
\] (3.2)
so that $\{h_n(x|q)\}$ is a $D_q$-Appell polynomial set and at the same time is an OPS satisfying the three term recurrence relation

$$
(1 - q^{n+1})h_{n+1}(x|q) = (1 - q)q^{n/2}xh_n(x|q) - \frac{1}{4}(1 - q)^2q^{-n/2}h_{n-1}(x|q), \tag{3.3}
$$

It also follows from (2.14) that any two polynomial sets $\{R_n(x)\}$ and $\{S_n(x)\}$, in that class are related by $R_n(x) = \sum_{k=0}^n c_{n-k}S_k(x)$. Thus the solution to our problem may be expressed as

$$
Q_n(x) = \sum_{k=0}^n a_{n-k}h_k(x|q), \tag{3.4}
$$

for some sequence of real constants $\{a_n\}$. We may assume without loss of generality that $a_0 = 1$.

The three term recurrence relation satisfied by $\{Q_n(x)\}$ is

$$
(1 - q^{n+1})Q_{n+1}(x) = \left((1 - q)q^{n/2}x + \beta_n\right)Q_n(x) - \gamma_nQ_{n-1}(x), \tag{3.5}
$$

with $Q_0(x) = 1$, $Q_1(x) = 0$. Thus $Q_1(x) = x + \beta_0 = a_1 + h_1(x|q)$, from which it follows that $a_1 = \beta_0$.

Putting (3.4) in (3.5) and using (3.3) to replace $xh_k(x|q)$ in terms of $h_{k+1}(x|q)$ and $h_{k-1}(x|q)$ we get, on equating coefficients of $h_k(x|q)$,

$$
(1 - q^{n+1})(1 + q^{n+1/2})a_{n+1} - \beta_n a_{n-1} + \left[\gamma_n - \frac{1}{4}(1 - q)^2q^{n+1/2}\right]a_{n-1} = 0, \tag{3.6}
$$

valid for all $n$ and $k = 0, 1, 2, ..., n + 1$ provided we interpret $a_{-1} = a_{-2} = 0$. It is easy to see that this system of equations is equivalent to the solution of our problem.

Putting $k = n$ in (3.6) we get

$$
\beta_n = (1 - q^{1/2})(1 + q^{n+1/2})a_1. \tag{3.7}
$$

Hence if $\beta_0 = 0$ then $\beta_n = 0$ for all $n$. In fact if $\beta_m = 0$ for any $n = m$ then $\beta_n = 0$ for all $n$.

Now we treat these two cases separately.

Case I. ($\beta_0 = 0$).

The system (3.6) can now be written as

$$
(1 - q^{k+1/2})(1 + q^{n+1/2})a_{k+1} + \left[\gamma_n - \frac{1}{4}(1 - q)^2q^{n+1/2}\right]a_{k-1} = 0. \tag{3.8}
$$

Since $a_1 = 0$ then it follows from (3.8) that $a_{2k+1} = 0$ for all $k$. In particular we get

$$
\gamma_n = \frac{1}{4}(1 - q)^2q^{-n/2} - a_2(1 - q)(1 + q^n), \tag{3.9}
$$

so that if $a_2 = 0$ then

$$
Q_n(x) = h_n(x|q). \tag{3.10}
$$

Now we show that $a_2 \neq 0$ leads to contradiction. To do this replace $k$ by $2k - 1$. We get

$$
(1 - q^k)(1 + q^{n-k+1})a_{2k} + \left[\frac{1}{4}(1 - q)^2q^{-n/2}(1 - q^{1-k}) - a_2(1 - q)(1 + q^n)\right]a_{2k-2} = 0. \tag{3.11}
$$

Keep $k$ fixed and let $n \to \infty$. We get $(1 - q^k)a_{2k} = (1 - q)a_2a_{2k-2}$. Thus

$$
a_{2k} = \frac{(1 - q)^k}{(q; q)_k}a_k^2. \tag{3.12}
$$

Putting this value in (3.11) we get $q^{1-k} = 1$. This is a contradiction and Case I is finished.

Case II ($\beta_0 \neq 0$).
We start with (3.6) we get, assuming \( a_1 \neq 0 \),

\[
\gamma_n = \frac{1}{4} (1 - q)^2 q^{n-\frac{3}{2}} + (1 - q^\frac{1}{2})(1 + q^{n+\frac{1}{2}})a_1^2 - (1 - q)(1 + q^n)a_2. \tag{3.13}
\]

Putting this value of \( \gamma_n \) and the value of \( \beta_n \) in (3.7) in (3.6), and finally equating coefficients of \( q^n \) and the terms independent of \( n \) we get the pair of equation systems

\[
(1 - q^{(k+1)/2})a_{k+1} - (1 - q^{\frac{1}{2}})a_1a_k + \left\{ (1 - q^{\frac{1}{2}})a_1^2 - (1 - q)a_2 \right\} a_{k-1} = 0 \tag{3.14}
\]

and

\[
(1 - q^{(k+1)/2})a_{k+1} - (1 - q^{\frac{1}{2}})q^{k/2}a_1a_k + \left\{ \frac{1}{4} (1 - q)^2 q^{-\frac{1}{2}} (q^{(k-1)/2} - 1) + q^{k/2} (1 - q^{\frac{1}{2}})a_1^2 - (1 - q)q^{(k-1)/2}a_2 \right\} a_{k-1} = 0 \tag{3.15}
\]

Eliminating \( a_{k+1} \) in these equations we get

\[
(1 - q^\frac{1}{2})(1 - q^{k/2})a_1a_k + \left\{ (1 - q)a_2, (1 - q^{(k-1)/2}) \right\} a_{k-1} = 0. \tag{3.16}
\]

This equation is of the form \((1 - q^{k/2})a_1a_k = c(1 - bq^{k/2})a_{k-1}\) so that the general solution of \((3.16)\) is

\[
a_k = c^k \frac{(by^\frac{1}{2}; q^\frac{1}{2})_k}{(y^\frac{1}{2}; q^\frac{1}{2})_k}. \tag{3.17}
\]

Putting this in (3.14) we get that \( b = 0 \). On the other hand (3.15) gives that \( c^2 = \frac{1}{4}(1 - q)^2 q^{-\frac{1}{2}} \). Finally putting those values of \( a_k \) in (3.13) we get that \( \gamma_n = 0 \) which is a contradiction.

This completes the proof of the theorem.

### 4. GENERATING FUNCTION

We obtain, for the q-Hermite polynomials, a generating function of the form (2.14). More specifically we prove

**Theorem 2.** Let \( H_n(x|q) \) be the \( n \)th Rogers q-Hermite polynomial. Then we have

\[
\sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}}{(q;q)_n} H_n(x|q) t^n = (t^2 q^{-\frac{1}{2}}; q^2)_\infty \mathcal{E}(x). \tag{4.1}
\]

**Proof.** Let \( A(t) = 1 + at + a_2t^2 + a_3t^3 + \cdots \) and

\[
A(t)\mathcal{E}(x) = \sum_{n=0}^{\infty} h_n(x|q) t^n. \tag{4.2}
\]

Then we get

\[
h_n(x|q) = \sum_{k=0}^{n} a_{n-k} c_k \Psi_k(x). \tag{4.3}
\]

where

\[
c_k = \frac{(1 - q)^k}{2^k (q; q)_k} q^{k(k-1)/4}. \tag{4.4}
\]
To calculate the coefficients \( \{a_n\} \) we first iterate (3.3) we get

\[
4x^2h_n(x|q) = \frac{4}{(1 - q)^2} (1 - q^{n+1})(1 - q^{n+2})q^{-\frac{1}{2}}h_{n+2}(x|q)
\]

(4.5)

\[
+ (2 - q^n - q^{n+1})h_n(x|q) + \frac{(1 - q)^2}{4} q^{-\frac{1}{2}}h_{n-2}(x|q).
\]

Putting (4.3) in (4.5), using (2.6) and then equating coefficients of \( \Psi_k(x) \) we get after some simplification

\[
4(1 - q^2)q^{-n-\frac{1}{2}}(1 - q^{-k+2})(1 - q^{n+k+1})a_{n+2-k} + \\
q^{-k-1} \left\{ 1 + q^{2k+2} - q^{n+k+1} - q^{n+k+2} \right\} a_{n-k} + \\
\frac{(1 - q)^2}{4} q^{-\frac{3}{2}} a_{n-2-k} = 0 \quad (k = 0, 1, \ldots, n + 2).
\]

(4.6)

By direct calculation of \( a_1, a_2, a_3 \) we see easily that \( a_1 = a_3 = 0 \). Thus (4.6) shows that \( a_{2k+1} = 0 \) for all \( k \).

Furthermore we can easily verify that

\[
a_{2j} = (-1)^j \frac{(1 - q)^{2j}}{(2q^2; q^2)_j} q^{(j-\frac{1}{2})} \quad (j = 0, 1, 2, 3, \ldots)
\]

(4.7)

Hence

\[
A(t) = \sum_{j=0}^{\infty} (-1)^j \frac{q^{j(j-1)}}{(q^2; q^2)_j} \left( \frac{(1 - q)^2j^2}{4} q^{-\frac{1}{2}} \right)^j
\]

(4.8)

\[
= \left( \frac{(1 - q)^2}{4} q^{-\frac{1}{2}} \right)^\infty.
\]

After some rescaling we get the theorem.

As a corollary of (4.1) we state the pair of inverse relations

\[
\Psi_n(x) = \sum_k \frac{(q; q)_n q^{(k-n)}(q^2; q^2)_k H_{n-2k}(x|q)}{(q^2; q^2)_k(q; q)_{n-2k}},
\]

(4.9)

\[
H_n(x|q) = \sum_k (-1)^k (q; q)_n \frac{q^{(2k-n-1)}(q^2; q^2)_k(q; q)_{n-2k}}{(q^2; q^2)_k(q; q)_{n-2k}} \Psi_{n-2k}(x).
\]

(4.10)

These follows from the identities (2.5) and (2.6)

Formula (4.10) and (2.11) give

\[
\frac{1}{e^{x^2} \left( \frac{1 - q^2}{4} q^{-\frac{1}{2}} D_8^2 \right)} \Psi_n(x).
\]

(4.11)

This is a q-analog of the formula

\[
e^{-D_8^2} x^n = H_n(x)
\]

for the regular Hermite polynomials (1.4).

References


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