A NOTE ON FINITE CODIMENSIONAL LINEAR ISOMETRIES OF $C(X)$ INTO $C(Y)$

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(Received April 24, 1994 and in revised form May 25, 1995)

ABSTRACT. Let $(X, Y)$ be a pair of compact Hausdorff spaces. It is shown that a certain property of the class of continuous maps of $Y$ onto $X$ is equivalent to the non-existence of linear isometry of $C(X)$ into $C(Y)$ whose range has finite codimension $> 0$.

KEY WORDS AND PHRASES. Compact Hausdorff space, $C(X)$, linear isometry, finite codimension

1991 AMS SUBJECT CLASSIFICATION CODES. 46B04, 46J10

1. INTRODUCTION

In [1], A. Gutke, D. Hart, J. Jamison and M. Rajagopalan proved that there are no isometric shift operators on $C([a, b])$, a result first proved in the real scalars by Holub [3]. Here $[a, b]$ is any closed interval in the real line and $C([a, b])$ is the Banach space of all continuous complex-valued functions on $[a, b]$. By observing carefully the proof given in [1], one can note that $C([a, b])$ does not admit an isometric shift operator because the space $[a, b]$ has the property that the set

\[ \{(x, y) \in [a, b] \times [a, b] : \phi(x) = \phi(y), x \neq y \} \]

is infinite for every continuous map $\phi$ of $[a, b]$ onto itself which is not injective.

The purpose of the note is to prove the following theorem which is based on the above idea:

THEOREM. Let $(X, Y)$ be a pair of compact Hausdorff spaces. Then the following two conditions are equivalent:

(i) If there is a continuous map $\phi$ of $Y$ onto $X$ which is not injective, then the set

\[ \{(y_1, y_2) \in Y \times Y : \phi(y_1) = \phi(y_2), y_1 \neq y_2 \} \]

is infinite.

(ii) If there is a linear isometry of $C(X)$ into $C(Y)$ which has a finite codimension, then it is surjective.

Since both $([0, 1], [0, 1])$ and $(T^1, T^1)$ satisfy the condition (i), where $T^1$ is the unit circle in the complex plane, we get from this:
COROLLARY 1. The only possible codimension of linear isometries $C([0, 1]) \rightarrow C([0, 1])$ and $C(T^1) \rightarrow C(T^1)$ are zero or infinite.

Moreover, if $V$ is the canonical linear map of $C(T^1)$ into $C([0, 1])$ defined by

$$(Vf)(t) = f(e^{2\pi it}) \quad (f \in C(T^1), 0 \leq t \leq 1),$$

then $V$ is an isometry and the range of $V$ is the set of all $g \in C([0, 1])$ such that $g(0) = g(1)$. Hence $V$ has codimension 1, and if there is a finite codimensional linear isometry of $C([0, 1])$ into $C(T^1)$, say $T$, then $VT$ is a linear isometry of $C([0, 1])$ into itself such that $VT(C([0, 1])) \subseteq C([0, 1])$ and $\text{codim}(T) + 1$. From Corollary 1 it follows that $VT$ must be surjective, a contradiction; hence we have also proved

COROLLARY 2. There is no finite codimensional linear isometry of $C([0, 1])$ into $C(T^1)$

2. LEMMAS

In order to prove the main theorem, we have to prepare some lemmas.

**LEMMA 1.** Let $X$ be a compact Hausdorff space, $M$ a subspace of $C(X)$ whose codimension is $n < +\infty$, and $K$ a closed boundary of $X$ with respect to $M$ (i.e., for any $f \in M$ there exists a point $x$ in $K$ with $|f(x)| = \|f\|_\infty$, the supremum norm of $f$ on $X$). Then the set $X\setminus K$ has at most $n$ points.

**PROOF.** Assume that $X\setminus K$ has at least $n + 1$ points, say $x_1, \ldots, x_{n+1}$. For each $1 \leq i \leq n + 1$, choose a function $f_i$ in $C(X)$ such that $f_i(x_i) = 1$ and $f_i(x) = 0$ for $x \in K \setminus \{x_1, \ldots, x_{n+1}\}$ since $K$ is closed. In this case, $\{f_1 + M, \ldots, f_{n+1} + M\}$ is linearly independent in $C(X)/M$ since

$$c_1(f_1 + M) + \cdots + c_{n+1}(f_{n+1} + M) = 0$$

for some complex numbers $c_1, \ldots, c_{n+1}$ there exists a function $g \in M$ such that $c_1f_1 + \cdots + c_{n+1}f_{n+1} + g = 0$ and (since $K$ is a boundary of $X$ with respect to $M$) a point $x_0$ in $K$ such that $\|g\|_\infty = |g(x_0)|$. Then

$$\|g\|_\infty = |c_1f_1(x_0) + \cdots + c_{n+1}f_{n+1}(x_0)| = 0,$$

implying $c_1 = 0, \ldots, c_{n+1} = 0$ since $\{f_1, \ldots, f_{n+1}\}$ is linearly independent, and it follows that $\text{codim}(M) \geq n + 1$.

**LEMMA 2.** Let $X$ and $Y$ be compact Hausdorff spaces and $\phi$ a continuous map of $Y$ onto $X$. If $g$ is a function in $C(Y)$ such that $g(y_1) = g(y_2)$ for all pairs $(y_1, y_2) \in Y \times Y$ satisfying $\phi(y_1) = \phi(y_2)$, then there is a function $f$ in $C(X)$ such that $f(\phi(y)) = g(y)$ for all $y \in Y$.

**PROOF.** Let $g$ be a function in $C(Y)$ such that $g(y_1) = g(y_2)$ for all pairs $(y_1, y_2) \in Y \times Y$ satisfying $\phi(y_1) = \phi(y_2)$. Let $Y/\phi$ be the quotient space of $Y$ defined by $\phi$, $\pi\circ\phi$ the canonical map of $Y$ onto $Y/\phi$, and $\tau$ the canonical map of $Y/\phi$ onto $X$. Then the complex-valued function $\tilde{g}$ on $Y/\phi$ defined by $\tilde{g}(\tilde{y}) = g(y)$ for each $\tilde{y} \in Y/\phi$ is continuous, so setting $f = \tilde{g} \circ \tau$ it is easy to see that $f$ is a function with the desired properties.

Finally, we will need the following result whose proof is straightforward.

**LEMMA 3.** Let $X$ be a compact Hausdorff space, $K$ a compact subset of $X$, and $A_K$ the Banach subspace of $C(X)$ consisting of all $f \in C(X)$ which are constant on $K$. Then the Banach space $C(X)/A_K$ is isomorphic to a quotient space of $C(K)$.

3. PROOF OF THEOREM

(i) $\Rightarrow$ (ii) Let $T$ be a linear isometry of $C(X)$ into $C(Y)$ which has a finite codimension. By the decomposition theorem of Holsztynski [2], there exists a closed boundary $K$ of $Y$ with respect to $T(C(X))$, a continuous map $h$ of $K$ onto $X$, and a continuous unimodular function $u$ on $Y$ such that

$$(Tf)(y) = u(y)f(h(y)).$$
for all \( f \in C(X) \) and \( y \in K \). Since \( T \) has a finite codimension, it follows from Lemma 1 that \( K \) is a closed subset of \( Y \) whose complement is a finite set. Then \( h \) has a continuous extension to \( Y \), say \( \tilde{h} \). We claim that the map \( \tilde{h} \) is injective. Assume the contrary. Then by the condition (i) there is a mutually different sequence \( \{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots\} \) in \( Y \) such that \( \tilde{h}(\alpha_n) = \tilde{h}(\beta_n) \) for all positive integers \( n \), and where we can assume without loss of generality that \( \{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots\} \subset K \). Let \( n \) be any positive integer, and for each \( 1 \leq i \leq n \) choose a function \( g_i \) in \( C(Y) \) such that \( g_i(\alpha_i) = 1 \) and \( g_i(y) = 0 \) for all \( y \in Y \setminus U_i \), where \( U_i \) is a sufficiently small neighborhood of \( \alpha_i \). In this case \( \{g_1 + T(C(X)), \ldots, g_n + T(C(X))\} \) is linearly independent in \( C(Y)/T(C(X)) \), since if
\[
c_1(g_1 + T(C(X))) + \ldots + c_n(g_n + T(C(X))) = 0
\]
for some complex numbers \( c_1, \ldots, c_n \) there exists \( f \in C(X) \) such that \( c_1 g_1 + \ldots + c_n g_n = T f \), implying
\[
c_i = c_1 g_1(\alpha_i) + \ldots + c_n g_n(\alpha_i)
= (Tf)(\alpha_i)
= u(\alpha_i) f(h(\alpha_i))
= u(\alpha_i) f(\tilde{h}(\beta_i))
= \frac{u(\alpha_i)}{u(\beta_i)} (Tf)(\beta_i)
= \frac{u(\alpha_i)}{u(\beta_i)} \{c_1 g_1(\beta_i) + \ldots + c_n g_n(\beta_i)\}
= 0
\]
for each \( i = 1, \ldots, n \). It follows that \( T \) has an infinite codimension since \( n \) is arbitrary, a contradiction. Consequently, \( \tilde{h} \) must be injective, \( K = Y \), and \( h \) is a homeomorphism of \( Y \) onto \( X \). If for any \( g \in C(Y) \), we set
\[
f(x) = \frac{1}{u(h^{-1}(x))} g(h^{-1}(x))
\]
for each \( x \in X \), then we obtain that \( f \in C(X) \) and \( T f = g \), so that \( T \) is surjective.

(ii) \( \Rightarrow \) (i). Let \( \phi \) be a continuous map of \( Y \) onto \( X \) which is not injective. Then we have to show that the set
\[
\{(y_1, y_2) \in Y \times Y : \phi(y_1) = \phi(y_2), y_1 \neq y_2\}
\]
is infinite under the condition (ii). If not, then all \( \phi^{-1}(x)(x \in X) \) are non-empty finite sets, and also \( \{x \in X : \text{card}(\phi^{-1}(x)) \geq 2\} \) is a non-empty finite set, say \( \{x_1, \ldots, x_n\} \), where "card" denotes the cardinal number. Set
\[
(T_\phi f)(y) = f(\phi(y))
\]
for each \( f \in C(X) \) and \( y \in Y \). Then \( T_\phi \) is a linear isometry of \( C(X) \) into \( C(Y) \) and since \( \phi \) is not injective, it follows that \( T_\phi \) is not surjective. Put
\[
A_i = \{g \in C(Y) : g \text{ is constant on } \phi^{-1}(x_i)\} \quad (i = 1, \ldots, n)
\]
and
\[
A = \left\{g \in C(Y) : g \text{ is constant on } \bigcup_{i=1}^n \phi^{-1}(x_i)\right\}.
\]
Then \( A \subseteq \bigcap_{i=1}^n A_i \), and hence \( C(Y)/\bigcap_{i=1}^n A_i \) is isomorphic to \( (C(Y)/A)/I \), where \( I = \{g + A \in C(Y)/A : g \in \bigcap_{i=1}^n A_i\} \). On the other hand, \( T_\phi(C(X)) = \bigcap_{i=1}^n A_i \), since the inclusion
\[ T_\phi(C(X)) \subseteq \bigcap_{i=1}^n A_i \] is trivial, and the reverse inclusion follows immediately from Lemma 2. Also by Lemma 3, \( C(Y)/A \) is isomorphic to a quotient of \( C(Y_0) \), where \( Y_0 = \bigcup_{i=1}^n \phi^{-1}(x_i) \). Consequently,

\[
\text{codim}(T_\phi) = \dim(C(Y)/T_\phi(C(X))) \\
\leq \dim(C(Y)/A) \\
\leq \dim(C(Y_0)) \\
\leq \sum_{i=1}^n \text{card}(\phi^{-1}(x_i)) \\
< +\infty.
\]

Hence \( T_\phi \) has a finite codimension, and so must be surjective by the condition (ii). But this is a contradiction, so the implication is proved.

**ACKNOWLEDGMENT.** The authors thank the referees for helpful comments and for improving the paper. The second author was partially supported by the Grant-in-Aid for Scientific Research from the ministry of Education, Science and Culture in Japan.

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