THE T-STATISTICALLY CONVERGENT SEQUENCES
ARE NOT AN FK-SPACE

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Abstract In this note we show that under certain restrictions on a nonnegative regular summability matrix $T$, the space of $T$-statistically convergent sequences cannot be endowed with a locally convex FK topology.

KEYWORDS AND PHRASES. Statistical convergence, FK-space

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1. INTRODUCTION

Statistical convergence, introduced by Fast [5], has most recently been studied by Fridy and Orhan [7] [8], and Kolk [9], among others [3] [4] [6] [11]. In [3], it is shown that the space of statistically convergent sequences cannot be endowed with a locally convex FK topology. In this note, we establish that under certain restrictions on a nonnegative regular summability matrix $T$, the space of $T$-statistically convergent sequences cannot be endowed with a locally convex FK topology.

An infinite matrix $T = (t_{nk})$ is nonnegative if $t_{nk} \geq 0$ for all $n$ and $k$, and regular if, for a convergent sequence $x$ with limit $l$, $\lim_n \sum_{k=1}^{\infty} t_{nk} x_k = l$. Throughout this note $T$ denotes a nonnegative regular matrix. We say that the rows of $T$ spread if $\lim_n \max_k t_{nk} = 0$. We let $\omega$ denote the space of all real valued sequences, $\varphi$ denote the finitely nonzero elements of $\omega$ and $N$ denote the positive integers. For $\epsilon > 0$ and a scalar $l$, we let $A_{\epsilon,l} = \{k : |x_k - l| < \epsilon\}$. A sequence $x \in \omega$ is $T$-statistically convergent to $l$ provided that for all $\epsilon > 0$,

$$\lim_n \sum_{k=1}^{\infty} t_{nk} \chi_{A_{\epsilon,l}}(k) = 1,$$

(1.1)

(where $\chi_A$ is the characteristic function of $A$). The space of $T$-statistically convergent sequences is denoted by $S_T$. Note that for $T = C_1$, the Cesàro matrix, this definition concurs with the definition of statistical convergence [6].

2. THE MAIN RESULT

A common theme in summability is the quest for “soft” methods to apply to classical type problems. An example of this is the “FK program,” in which a summability space is given an FK topology ([13], pg. 54). An FK space $X$ is a subspace of $\omega$ with a complete locally convex Fréchet topology such that the inclusion map from $X$ into $\omega$ is continuous [13]. Our result shows that we cannot apply the FK-program to $T$-statistical convergence and improves Theorem 3.3 of [3].
THEOREM 1. For a nonnegative regular summability matrix $T$ whose rows spread, the space of $T$-statistically convergent sequences cannot be endowed with a locally convex FK topology.

The proof of this theorem depends upon the following result of Bennett and Kalton [2].

THEOREM 2. Let $S$ be a dense subspace of $\omega$. Then the following are equivalent:
1) $S$ is barrelled.
2) If $E$ is a locally convex FK space that contains $S$, then $E = \omega$.

Note that the above statement is a restricted version of the result in [2], and an exposition can be found in ([12], pg. 253).

The proof of the main result follows that of Theorem 3.3 in [3].

PROOF OF THEOREM 1:

We show that $S_T$ is a dense barrelled subspace of $\omega$. Recall that $S_T$ is barrelled if and only if every $\sigma(\varphi, S_T)$-bounded subset of $\varphi$ is $\sigma(\varphi, \omega)$-bounded ([12], pg.248). Thus, to show $S_T$ is barrelled it suffices to show that if $E$ is not $\sigma(\varphi, \omega)$-bounded, then $E$ is not $\sigma(\varphi, S_T)$-bounded.

We may assume that $E$ is $\sigma(\varphi, \varphi)$-bounded, since otherwise $E$ is not $\sigma(\varphi, S_T)$-bounded and we are done. Thus, there exists a sequence of integers $< B_n >$ such that for $n$ an element of $E$ with $\text{spt}(x) = \{k \in \mathbb{N} : x_k \neq 0\} \subseteq \{1, 2, \ldots, n\}$, we have $\sup_{1 \leq i \leq n} |x_i|$ is less than or equal to $B_n$. Since $E$ is not $\sigma(\varphi, \omega)$-bounded, we can choose $s$ in $\omega$ such that $\sup_{z \in E} |\sum_{i=1}^{n} x_i s_i|$ is infinite.

Note that for all $z$ in $E$, we have $|\sum_{i=1}^{n} x_i s_i| \leq nB_n \sup_{1 \leq i \leq n} |s_i|$.

Select $x^1 \in E$ such that $|\sum_{i=1}^{\infty} x^1_i s_i| > B_1 |s_1|$, and select $j_1 > 1$ such that $x^1_{j_1} \neq 0$ (such a $j_1$ exists since $E$ is not $\sigma(\varphi, \omega)$-bounded and since $|x_1| \leq B_1$). Assume that $\{x^1, x^2, \ldots, x^n\}$ and $j_1 < j_2 < \ldots < j_n$ have been chosen so that $x^n_{j_n} \neq 0$ and $j_n > \max\{k \in \mathbb{N} : k \in \text{spt}(x^n)\}$. Set $t = \max\{j_n, \max\{k \in \mathbb{N} : k \in \text{spt}(x^n)\}\}$, and select $x^{n+1}$ such that

$$|\sum_{i=1}^{\infty} x_i^{n+1} s_i| > tB_1 \sup_{k \leq t} |s_k|.$$  \hspace{1cm} (2.1)

Now select $j_{n+1}$ such that $t < j_{n+1}$ and $x_{j_{n+1}}^{n+1} \neq 0$, and proceed inductively.

By [10], since the rows of $T$ spread there is a subsequence $< j_{p_m} >$ of $< j_n >$ such that $\lim_{m \to \infty} \sum_{k=1}^{\infty} t_k \chi_{\{m \in \mathbb{N}\}}(k) = 0$. Since $x_k^{p_m} \neq 0$ for all $m$, it is possible to construct a sequence $\alpha = (\alpha_k)$ such that $\sum_{k=1}^{\infty} \alpha_k x_k^{p_m} \to \infty$ as $m \to \infty$. Then we set

$$z = (z_r) = \begin{cases} 
\alpha_k & \text{if } r = j_{p_k}, \text{ for } k = 1, 2, \ldots \\
0 & \text{else.}
\end{cases}$$  \hspace{1cm} (2.2)

Now, $z$ is $T$-statistically convergent to 0 (because the non-zero entries of $z$ occur on the subsequence $< j_{p_m} >$ and by the definition of $T$-statistical convergence). Note also that $\sum_{k=1}^{\infty} x_k^{p_m} z_k = \sum_{k=1}^{\infty} x_k^{p_m} \alpha_k$. Since the right hand side of this equation tends to infinity as $m$ does, it follows that $E$ is not $\sigma(\varphi, S_T)$-bounded. Since $\varphi \subseteq S_T$, $S_T$ is a dense barrelled subspace of $\omega$. Now, by the result of Bennett and Kalton, we have that $S_T$ cannot be endowed with a locally convex FK topology.

The following examples illustrate the necessity of the hypothesis that the rows of $T$ spread. Consider the identity matrix $I = (i_{nk})$, where $i_{nn} = 1$ and $i_{nk} = 0$ for all $k \neq n$. The space
of \( I \)-statistically convergent sequences is the space of convergent sequences \( c \), a well-known FK space. For a less trivial example, consider the matrix \( T = (t_{nk}) \) where \( t_{11} = 1 \), \( t_{1k} = 0 \) for \( k \geq 2 \), and for \( n \geq 2 \),

\[
t_{nk} = \begin{cases} 
\frac{1}{2} & \text{if } k = n \text{ or } k = n - 1, \\
0 & \text{else.}
\end{cases}
\]

Note that the rows of \( T \) spread, and that this method is regular. In fact, \( T \) is stronger than convergence since the sequence \( x = \langle (-1)^n \rangle \) has \( \lim_{n} \sum_{k=1}^{\infty} t_{nk} x_k = 0 \). As in the case of the identity matrix, the \( T \)-statistically convergent sequences are again the FK space \( c \) of convergent sequences.

A consequence of this result is that we cannot employ the FK program when studying \( T \)-statistical convergence for a matrix \( T \) whose rows spread. Instead, the Stone-Čech compactification of the integers has been used [4][1] as an avenue for “soft” methods for treating \( T \)-statistical convergence of bounded sequences. This result also includes Corollary 4.4 of [9], where it is shown that under certain restrictions, a matrix \( B \) maps the space of \( T \)-statistically convergent sequences into a sequence space \( Y \) if and only if \( B \) has at most finitely many non-zero columns which belong to \( Y \).

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