RESEARCH NOTES

TWO INEQUALITIES FOR MEANS

J. SANDOR

4160 Forteni Nr. 79
R-Jud. Harghita
Romania

(Received September 22, 1993 and in revised form February 22, 1994)

ABSTRACT. We prove two new inequalities for the identric mean and a mean related to the arithmetic and geometric mean of two numbers.

KEY WORDS AND PHRASES. Identric mean, logarithmic mean, quadrature formulas.

1991 AMS SUBJECT CLASSIFICATION CODES. 26D99, 65D32.

1. INTRODUCTION.

The logarithmic and identric means of two positive numbers \(a\) and \(b\) are defined by

\[
L = L(a, b) = \frac{\log b - \log a}{\log b - \log a} \quad \text{for} \quad a \neq b; \quad L(a, a) = a
\]

and

\[
I = I(a, b) = \frac{b^a/a^b}{(b-a)^{1/(b-a)}} \quad \text{for} \quad a \neq b; \quad I(a, a) = a,
\]

respectively.

Let \(A = A(a, b) = \frac{a+b}{2}\) and \(G = G(a, b) = \sqrt{ab}\) denote the arithmetic and geometric means of \(a\) and \(b\), respectively. Many interesting results have been proved for these means, see e.g. ([1] - [3], [5] - [10]). Let us introduce the mean \(U\) defined by

\[
U = U(a, b) = \left(\frac{(2a+b)(a+2b)}{9}\right)^{1/2} = \left(\frac{8A^2 + G^2}{9}\right)^{1/2}
\]

The aim of this note is to prove the following:

THEOREM. For \(a \neq b\) one has

\[
(U^3G)^1/4 < I < \frac{U^2}{A}.
\] (1.1)

2. PROOF OF THE THEOREM.

For the first inequality we apply the Newton quadrature formula (see [4])

\[
\int_a^b f(x)\,dx = \frac{b-a}{8} \left[ f(a) + 3f\left(\frac{a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{(b-a)^5}{640} f^{(4)}(\xi),
\] (2.1)

where \(\xi \in (a, b)\) and \(f: [a, b] \to \mathbb{R}\) has a continuous 4-th derivative on \((a, b)\). Let \(f(x) = -\log x (x > 0)\) in (2.1). Then \(f^{(4)}(x) > 0\), and after certain transformations we get the left side of 1.1.
In order to prove the second inequality of (1.1) divide all terms by \( a < b \) and denote \( x = \frac{b}{a} > 1 \). Then the inequality to be proved becomes

\[
(4x^2 + 10x + 4)/(x + 1)g(x) > 9/e \tag{2.2}
\]

where \( g(x) = x^{p/2-1}, \ x > 1 \).

Introduce the function \( f : [1, \infty) \rightarrow \mathbb{R} \) defined by

\[
f(x) = (4x^2 + 10x + 4)/(x + 1)g(x), \ x > 1; \ f(1) = \lim_{x \to 1} f(x) = 9/e
\]

We shall prove that \( f \) is strictly increasing, and this proves (2.2). We have

\[
g'(x) = g(x) \left[ \frac{1}{x-1} - \frac{\log x}{(x-1)^2} \right],
\]

and, after some elementary computations, we can deduce

\[
(x^2 - 1)^2 g(x)f'(x) = (4x^2 + 10x + 4)(x + 1)\log x - 10x^3 - 6x^2 + 6x + 10 \tag{2.3}
\]

We now show that the right side of (2.3) is strictly positive, or equivalently

\[
L < (8A^2 + G^2)A/(10A^2 + G^2) \tag{2.4}
\]

where \( L = L(x, 1) \) etc. Since it is known that \( L < (2G + A)/3 \) (See [3]) we try to prove that \((2G + A)/3 < (8A^2 + G^2)A/(10A^2 - G^2)\). This holds true iff \(14x^3 - 20x^2y + 4xy^2 + 2y^3 > 0\), with \( x = A, y = G \), i.e.,

\[
(x - y)(7x^2 - 3xy - y^2) > 0 \tag{2.5}
\]

We have

\[
7x^2 - 3xy - y^2 = \left[ x + y \left( \frac{\sqrt{37} - 3}{14} \right) \right] \left[ x - y \left( \frac{\sqrt{37} + 3}{14} \right) \right] > 0 \text{ by } \frac{\sqrt{37} - 3}{14} > 0
\]

and \(0 < \frac{\sqrt{37} + 3}{14} < 1\). Thus (2.5) is proved, concluding the proof of (2.2) and of the theorem.

3. REMARKS.

(1) Clearly, \( G < U < A \) (for \( a \neq b \)). Relation (1.1) offers the improvement

\[
G < (U^3G)^{1/4} < I < U^2 < U < A \tag{2.6}
\]

(2) It is well-known that (see e.g. [7]) \( A > I \), so from the right inequality in (1.1) we have

\[
9I^2 < 8A^2 + G^2 \tag{2.7}
\]

On the other hand, it is known that \([8] \ I > (2A + G)/3\), which according to \( A > G \) and (2.7) yields the following double-inequality:

\[
4A^2 + 5G^2 < 9I^2 < 8A^2 + G^2 \tag{2.8}
\]

(3) The two sides of (1.1) imply

\[
U^5 > A^4G \tag{2.9}
\]

ACKNOWLEDGEMENTS. The author wishes to thank the referee for several useful remarks.
REFERENCES