A COMMON FIXED POINT THEOREM FOR A SEQUENCE OF FUZZY MAPPINGS

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(Received September 29, 1994)

ABSTRACT. We obtain a common fixed point theorem for a sequence of fuzzy mappings, satisfying a contractive definition more general than that of Lee, Lee, Cho and Kim [2].

Let \((X,d)\) be a complete linear metric space. A fuzzy set \(A\) in \(X\) is a function from \(X\) into \([0,1]\). If \(x \in X\), the function value \(A(x)\) is called the grade of membership of \(x\) in \(A\). The \(\alpha\)-level set of \(A\), \(A_\alpha := \{x : A(x) \geq \alpha\}\), if \(\alpha \in (0,1]\), and \(A_0 := \{x : A(x) > 0\}\). \(W(X)\) denotes the collection of all the fuzzy sets \(A\) in \(X\) such that \(A_\alpha\) is compact and convex for each \(\alpha \in [0,1]\) and \(\sup_{x \in X} A(x) = 1\). For \(A, B \in W(X), A \subset B\) means \(A(x) \leq B(x)\) for each \(x \in X\). For \(A, B \in W(X), \alpha \in [0,1]\), define

\[
\begin{align*}
P_\alpha(A,B) &= \inf_{x \in A_\alpha, y \in B_\alpha} d(x,y), \\
P(A,B) &= \sup_\alpha P_\alpha(A,B), \\
D(A,B) &= \sup_\alpha d_H(A_\alpha, B_\alpha),
\end{align*}
\]

where \(d_H\) is the Hausdorff metric induced by the metric \(d\). We note that \(P_\alpha\) is a nondecreasing function of \(\alpha\) and \(D\) is a metric on \(W(X)\).

Let \(X\) be an arbitrary set, \(Y\) any linear metric space. \(F\) is called a fuzzy mapping if \(F\) is a mapping from the set \(X\) into \(W(Y)\).

In earlier papers the author and Bruce Watson, [3] and [4], proved some fixed point theorems for some mappings satisfying a very general contractive condition. In this paper we prove a fixed point theorem for a sequence of fuzzy mappings satisfying a special case of this general contractive condition. We shall first prove the theorem, and then demonstrate that our definition is more general than that appearing in [2].

Let \(D\) denote the closure of the range of \(d\). We shall be concerned with a function \(Q\), defined on \(D\) and satisfying the following conditions:

(a) \(0 < Q(s) < s\) for each \(s \in D \setminus \{0\}\) and \(Q(0) = 0\),

(b) \(Q\) is nondecreasing on \(D\), and

(c) \(g(s) := s / (s - Q(s))\) is nonincreasing on \(D \setminus \{0\}\).

**Lemma 1.** [1] Let \((X,d)\) be a complete linear metric space, \(F\) a fuzzy mapping from \(X\) into \(W(X)\) and \(x_0 \in X\). Then there exists an \(x_1 \in X\) such that \(\{x_1\} \subset F(x_0)\).

**KEY WORDS AND PHRASES:** Common fixed point, fuzzy mappings.

THEOREM 1. Let $g$ be a nonexpansive selfmap of $X$, $(X,d)$ a complete linear metric space. Let $\{F_i\}$ be a sequence of fuzzy mappings from $X$ into $W(X)$ satisfying: For each pair of fuzzy mappings $F_i, F_j$ and for any $x \in X, \{u_x\} \subset F_i(x)$, there exists a $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that

$$D(\{u_x,v_y\}) \leq Q(m(x,y)),$$

where

$$m(x,y) := \max\{d(g(x),g(u_x)), d(g(y),g(v_y)), \left[ d(g(y),g(v_y)) + d(g(x),g(u_x)) \right]/2, d(g(x),g(y)) \}$$

and $Q$ satisfies (a) - (c). Then there exists a $p \in \cap_{i=1}^{\infty} F_i(p)$.

PROOF. Let $x_0 \in X$. Then we can choose $x_1 \in X$ such that $\{x\} \subset F_1(x_0)$ by Lemma 1. From the hypothesis, there exists an $x_2 \in X$ such that $\{x_2\} \subset F_2(x_1)$ and, from (1),

$$D(\{x_1,x_2\}) \leq Q(m(x_1,x_2))$$

$$< \max\{d(g(x_1),g(x_2)), d(g(x_1),g(x_2)) \}$$

$$\leq \max\{d(x_0,x_1), d(x_1,x_2), [d(x_1,x_1) + d(x_0,x_2)]/2, d(x_0,x_1) \},$$

since $g$ is nonexpansive.

Inductively, we obtain a sequence $\{x_n\}$ such that $x_{n+1} \subset F_{n+1}(x_n)$ and

$$D(\{x_n, x_{n+1}\}) \leq Q(m(x_n,x_{n+1}))$$

$$< \max\{d(x_{n-1},x_n), d(x_n,x_{n+1}), [d(x_n,x_n) + d(x_{n-1},x_{n+1})]/2, d(x_{n-1},x_n) \},$$

(3)

Since $D(\{x_n, x_{n+1}\}) = d(x_n,x_{n+1})$, it follows from (3) that $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$. Using this fact back in (2), we obtain that $d(x_n,x_{n+1}) \leq d(x_{n-1},x_n)$. Substituting into (3) we obtain

$$d(x_n,x_{n+1}) < Q(d(x_{n-1},x_n)) < Q^2(d(x_{n-2},x_{n-1}) < \cdots < Q^n(d(x_0,x_1)).$$

From Lemma 2 of [3], $\lim Q^n(d(x_0,x_1)) = 0$. To show that $\{x_n\}$ is Cauchy, choose $N$ so large that $Q^n(d(x_0,x_1)) \leq 1/2$ for all $n > N$. Then, for $m > n > N$,

$$d(x_n,x_m) \leq \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \leq \sum_{i=n}^{m-1} Q^i(d(x_0,x_1)) \leq \sum_{i=n}^{m-1} \left( \frac{1}{2} \right)^i < \frac{1}{2^n} \left( 1 - \frac{1}{2} \right),$$

and $\{x_n\}$ is Cauchy, hence convergent. Call the limit $p$.

Let $F_m$ be an arbitrary member of the sequence $\{F_i\}$. Since $\{x_n\} \subset F_m(x_{n-1})$, there exists a $v_n \in X$ such that $\{v_n\} \subset F_m(p)$ for all $n$ and

$$D(\{x_n, v_n\}) \leq Q(m(x_n,v_n))$$

$$= Q\left( \max\{d(g(x_{n-1}),g(x_n)), d(g(p),g(v_n)) \} \right)$$

$$\leq Q\left( \max\{d(x_{n-1},x_n), d(p,v_n), [d(x_{n-1},v_n) + d(p,x_n)]/2, d(x_{n-1},p) \} \right).$$

Suppose that $\lim v_n \neq p$. Taking the limit as $n \to \infty$ yields, since $Q$ is continuous (Lemma 1 of [3]),

$$\limsup d(p,v_n) \leq Q(\limsup d(p,v_n)) < \lim d(p,v_n),$$

which is a contradiction. Therefore, $\lim v_n = p$. Thus, $\{x_n\}$ converges to $p$.
a contradiction. Therefore \( \lim v_n = p \). Since \( F_m(p) \in W(X) \), \( F_m(p) \) is upper semicontinuous and therefore \( \limsup [F_m(p)](v_n) \leq [F_m(p)](p) \). Since \( \{v_n\} \subset F_m(p) \) for all \( n, [F_m(p)](p) = 1 \). Hence \( \{p\} \subset F_m(p) \). Since \( F_m \) is arbitrary, \( \{p\} \subset \cap_{m=1}^{\infty} F_m(p) \).

The contractive definition of [2] is the following:

\[
D(\{u_x\},\{v_y\}) \leq a_1 d(g(x),g(u_x)) + a_2 d(g(y),g(v_y)) + a_3 d(g(y),g(u_x)) + a_4 d(g(x),g(y)) \\
+a_5 d(g(x),g(v_y)) + d(g(x),g(y)) \\
(4)
\]

where each \( a_i \) is nonnegative, \( \sum_{i=1}^{5} a_i < 1 \), and \( a_3 \geq a_4 \).

In (4), if one interchanges the roles of \( x \) and \( y \) one obtains

\[
D(\{v_y\},\{u_x\}) \leq a_1 d(g(y),v_y) + a_2 d(g(x),g(u_x)) + a_3 d(g(y),g(u_x)) + \\
a_4 d(g(y),g(u_x)) + a_5 d(g(y),g(x)) .
(5)
\]

Adding (4) and (5) yields

\[
D(\{u_x\},\{v_y\}) \leq a_1 d(g(x),g(u_x)) + a_2 d(g(y),g(v_y)) + a_3 d(g(y),g(u_x)) + \\
a_4 d(g(x),g(v_y)) + a_5 d(g(x),g(y)),
(6)
\]

where \( \alpha_1 = \alpha_2 = (a_1 + a_2)/2, \alpha_3 = a_4 = (a_3 + a_4)/2, \) and \( \alpha_5 = a_5 \). In turn, (6) implies that

\[
D(\{u_x\},\{v_y\}) \leq (\alpha_1 + \alpha_2) \max \{d(g(x),g(u_x)), d(g(y),g(v_y))\} + \\
\alpha_3 [d(g(y),g(v_y)) + d(g(x),g(v_y))] / 2 + \alpha_5 d(g(x),g(y)) \\
\leq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5) m(x,y) = \left( \sum_{i=1}^{5} a_i \right) m(x,y) = h m(x,y),
(7)
\]

say.

(7) is the special case of (1) with \( Q(s) = hs \). Consequently Theorem 3.1 of [2], as well as the corollaries, are special cases of the theorem of this paper.

REFERENCES


