ABSTRACT. For a monotone convex function $f \in C[a,b]$ we prove that the modulus of continuity $\omega(f;h)$ is concave on $[a,b]$ as function of $h$. Applications to approximation theory are obtained.

KEY WORDS AND PHRASES. Concave modulus of continuity, approximation by positive linear operators, Jackson estimate in Kornechuk's form.

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1. INTRODUCTION.

In a recent paper, Gal [1] the modulus of continuity for convex functions is exactly calculated, in the following way.

THEOREM 1. (see [1]) Let $f \in C[a,b]$ be monotone and convex on $[a,b]$. For any $h \in [0,b-a]$ we have:

(i) $\omega(f;h) = f(b) - f(b-h)$, if $f$ is increasing on $[a,b]$,

(ii) $\omega(f;h) = f(a) - f(a + h)$, if $f$ is decreasing on $[a,b],

where $\omega(f;h)$ denotes the classical modulus of continuity.

Denote

$KM[a,b] = \{f \in C[a,b]; \text{ f is monotonous convex or monotonous concave on } [a,b]\}$

The purpose of the present paper is to prove that for $f \in KM[a,b]$ the modulus of continuity $\omega(f;h)$ is concave as function of $h \in [0,b-a]$ and to apply this result to approximation by positive linear operators and to Jackson estimates in Kornechuk's form.

2. MAIN RESULTS AND APPLICATIONS.

A first main result is the following

THEOREM 2. For all $f \in KM[a,b]$, the modulus of continuity $\omega(f;h)$ is concave as function of $h \in [0,b-a]$.

PROOF. Let firstly suppose that $f$ is increasing and convex on $[a,b]$. If $f$ is increasing on $[a,b]$, by Theorem 1, (i), we have $\omega(f;h) = f(b) - f(b-h)$. Hence

$$\omega(f;h) = f(b) - f(b-h)$$

for all $h \in [0,b-a]$. 

\[\begin{aligned}
\alpha \omega(f;h_1) + (1 - \alpha) \omega(f;h_2) &= f(b) - \alpha f(b - h_1) - (1 - \alpha) f(b - h_2) \\
\omega(f;\alpha h_1 + (1 - \alpha)h_2) &= f(b) - f(b - \alpha h_1 - (1 - \alpha)h_2)
\end{aligned}\] (1.1) (1.2)

for all $h_1, h_2 \in [0,b-a]$. 

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\omega(f;\alpha h_1 + (1 - \alpha)h_2) &= f(b) - f(b - \alpha h_1 - (1 - \alpha)h_2)
\end{aligned}\] (1.1) (1.2)

for all $h_1, h_2 \in [0,b-a]$.
Since $f$ is convex on $[a,b]$ we get
\[ f(b - ah_1 - (1 - a)h_2) \leq af(b - h_1) + (1 - a)f(b - h_2), \]
wherefrom taking into account (1.1) and (1.2) too, we get
\[ \alpha \omega(f;h_1) + (1 - \alpha)\omega(f;h_2) \leq \omega(f;ah_1 + (1 - a)h_2) \tag{1.3} \]

Now, if $f$ is decreasing on $[a,b]$, since by Theorem 1, (ii), we have $\omega(f;h) = f(a) - f(a + h)$, we immediately get
\[ \alpha \omega(f;h_1) + (1 - \alpha)\omega(f;h_2) = f(a) - af(a + h_1) - (1 - \alpha)f(a + h_2) \tag{1.4} \]
and
\[ \omega(f;ah_1 + (1 - a)h_2) = f(a) - f(a + ah_1 + (1 - a)h_2) \tag{1.5} \]
for all $\alpha \in [0,1]$ and all $h_1, h_2 \in [0, b - a]$.

Since $f$ is convex on $[a,b]$ we have
\[ f(a + ah_1 + (1 - a)h_2) \leq af(a + h_1) + (1 - \alpha)f(a + h_2), \]
which together with (1.4) and (1.5) gives again (1.3).

In the following we need the

**DEFINITION 1.** (see e.g. [2]) Let $f \in C[a,b]$ be. If $\omega(f;h) = \sup \{ |f(x) - f(y)| : |x - y| \leq h \}$ is the usual modulus of continuity, the least concave majorant of $\omega(f;h)$ is given by
\[ \overline{\omega}(f;h) = \sup \left\{ (\alpha)\omega(f;\beta) + (\beta - \alpha)\omega(f;\alpha) : 0 \leq \alpha \leq \beta \leq b - a \right\}. \]

An immediate consequence of Definition 1 is the

**COROLLARY 1.** For any $f \in KM[a,b]$ we have
\[ \overline{\omega}(f;h) = \omega(f;h) \]

**PROOF.** Putting $\alpha = \delta$ in Definition we get
\[ \omega(f;h) \leq \overline{\omega}(f;h). \]
Then, taking into account Theorem 2, for $0 \leq \alpha \leq \delta \leq \beta \leq b - a$ we have
\[ \frac{(\delta - \alpha)\omega(f;\beta) + (\beta - \delta)\omega(f;\alpha)}{\beta - \alpha} \leq \omega \left( f; \frac{\beta(\delta - \alpha)}{\beta - \alpha} + \frac{\alpha(\beta - \delta)}{\beta - \alpha} \right) = \omega(f;\delta) \]
wherefrom passing to supremum, we immediately get
\[ \overline{\omega}(f;\delta) \leq \omega(f;\delta), \]
which proves the corollary.

**REMARK.** It is easy to see that Corollary 1 remains valid for all $f \in C[a,b]$ having a concave modulus of continuity $\omega(f;h)$.

Now, firstly we will apply the previous results to approximation by positive linear operators.

Thus, investigating the sequence of Lehnhoff polynomials in [3], $L_n(f)(x)$, defined for $f \in C[-1,1]$, H.H. Gonska [2] proves that
\[ |L_n(f)(x) - f(x)| \leq \frac{\sqrt{30}}{\sqrt{n}} \left( f; \sqrt{\frac{1 - x^2}{n}} + \frac{|x|}{n} \right) \]
Taking now into account Corollary 1 we immediately get the
COROLLARY 2. If $f \in K_M[-1,1]$ then for all $x \in [-1,1], n \in N$ we have
\[ |I_n(f)(x) - f(x)| \leq \sqrt{\frac{10}{n}} \left( f; \frac{\sqrt{1 - x^2}}{n} + \frac{|x|}{n^2} \right) \]

In the same paper, for $f \in C[0,1]$, H.H. Gonska obtains estimates in terms of the modulus $(f;h)$ in the approximation by the so-called Shepard operator, $S_n^h(f), 1 \leq \mu \leq 2$. Then by Corollary 1 and by Theorem 4.3 in [2] we immediately get the

COROLLARY 3. For all $f \in K_M[0,1]$ and all $n \in N$ we have
\[ \|S_n^f(f) - f\| \leq \frac{n + 1}{n} \omega(f; \frac{1}{(n+1)(n+2)}) \]
\[ \|S_n^h(f) - f\| \leq \frac{14}{2 - \mu} \omega(f; \frac{1}{(n+1)(n+2)}) \]
\[ \|S_n^2(f) - f\| \leq 19\omega(f; \frac{1}{n+1}) \]

Finally, we will apply our results to the following so-called Jackson estimate in Korneichuk’s form.

THEOREM 3. (see e.g. [4], p. 147) For any $f \in C[-1,1]$ we have
\[ E_n(f) \leq \omega(f; \frac{1}{n}), n = 1, 2, \cdots \]
where $E_k(f)$ denotes the best approximation by polynomials of degree $\leq k$.

Now, we will prove the

THEOREM 4. If $f \in C[-1,1]$ has a concave modulus of continuity $\omega(f;h), h \in [0,2]$, then we have
\[ E_n(f) \leq \frac{1}{2} \omega(f; \frac{\pi}{n}) \]

PROOF. Extending $\omega$ to $[0,\pi]$ by taking $\omega(f;h) = \omega(f;2), h \in [2,\pi]$, obviously $\omega$ remains concave on $[0,\pi]$.

Denote $\omega(h) = \omega(f;h), h \in [0,\pi]$ and
\[ \Lambda_\omega = \{g \in C[-1,1]; \omega(g;h) \leq \omega(h), \forall h \in [0,\pi]\} \]

Obviously $f \in \Lambda_\omega$. Then by [5, Theorem 8 and Lemma 2, p. 122-123], as in the proof of Theorem 9, p. 123 in [5], there is $g \in Lip_M^1$ such that
\[ \|f - g\| \leq \frac{1}{2} \omega(f; \frac{\pi}{2n}) \]

Now by Theorem V, (ii), in [4, p. 147], there is $P_{n-1}$ polynomial of degree $\leq n-1$ such that
\[ \|g - P_{n-1}\| \leq \frac{\pi M}{2n}. \]

Hence we get
\[ \|f - P_{n-1}\| \leq \|f - g\| + \|g - P_{n-1}\| \leq \frac{1}{2} \omega(f; \frac{\pi}{n}) \]

which proves the theorem.

REMARK. For $f \in K_M[-1,1]$, Theorem 4 remains valid.
REFERENCES


