EXISTENCE OF PERIODIC SOLUTIONS
FOR NONLINEAR LIENARD SYSTEMS

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ABSTRACT. We prove the existence and multiplicity of periodic solutions for nonlinear Lienard System of the type

\[ x''(t) + \frac{d}{dt} [\nabla F(x(t))] + g(x(t)) + h(t, x(t)) e(t) = 0 \]

under various conditions upon the functions \( g, h \) and \( e \).

KEY WORDS AND PHRASES: Nonlinear Lienard system, multiplicity of periodic solution.

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1. INTRODUCTION

Let \( \mathbb{R}^n \) be \( n \)-dimensional Euclidean space. We define \( x_{[a,b]} = [x_1, x_2, \ldots, x_n] \) for \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \). By \( L^2([0, 2\pi], \mathbb{R}^n) \) we denote the space of all measurable functions \( x: [0, 2\pi] \to \mathbb{R}^n \) for which \( \int_0^{2\pi} \|x(t)\|^2 dt \) is integrable. The norm is given by

\[ \|x\|_{L^2} = \left[ \int_0^{2\pi} \|x(t)\|^2 dt \right]^{1/2}. \]

By \( C^k([0, 2\pi], \mathbb{R}^n) \) we denote the Banach space of \( 2\pi \)-periodic continuous functions \( x: [0, 2\pi] \to \mathbb{R}^n \) whose derivatives up to order \( k \) are continuous. The norm is given by

\[ \|x\|_{C^k} = \sum_{i=0}^{k} \|x^{(i)}\|_{L^2} \]

where \( \|y\| = \sup_{t \in [0, 2\pi]} \|y(t)\| \) is a norm in \( C([0, 2\pi], \mathbb{R}^n) \). We use the symbol \( \langle \cdot, \cdot \rangle \) for the Euclidean inner product in the space \( \mathbb{R}^n \). For \( x, y \) in \( C([0, 2\pi], \mathbb{R}^n) \) we define the \( L^2 \)-inner product as follows

\[ \langle x, y \rangle = \int_0^{2\pi} (x(t), y(t)) dt. \]

The mean value \( \bar{x} \) of \( x \) and the function of mean value zero are defined by \( \bar{x} = \frac{1}{2\pi} \int_0^{2\pi} x(t) dt \) and \( \bar{y}(t) = x(t) - \bar{x}, \) respectively.

We define inequalities in \( \mathbb{R}^n \) componentwise, i.e. \( x, y \in \mathbb{R}^n, \ x \leq y \) if and only if \( x_i \leq y_i \) for \( i = 1, 2, \ldots, n, \) and \( x < y \) if and only if \( x_i < y_i \) for \( i = 1, 2, \ldots, n. \) In this work, we will study the existence of periodic solutions and multiple periodic solutions for the problem

\[ x''(t) + \frac{d}{dt} [\nabla F(x(t))] + g(x(t)) + h(t, x(t)) e(t) = 0 \]

\[ x(0) = x(2\pi), \ x'(0) = x'(2\pi) = 0 \]
where \( F : \mathbb{R}^n \rightarrow \mathbb{R} \) is a \( C^2 \)-function, \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) is continuous, \( h : [0, 2\pi] \times \mathbb{R}^n \rightarrow \mathbb{R} \) is continuous in both variables and 2\( \pi \)-periodic in \( t \), and \( e : [0, 2\pi] \rightarrow \mathbb{R} \) is in \( L^2([0, 2\pi], \mathbb{R}^n) \). We assume that \( g(x) = (g_1(x), g_2(x), \ldots, g_n(x)) \) for all \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( h(t, x) = (h_1(t, x), h_2(t, x), \ldots, h_n(t, x)) \) for all \( (t, x) \in [0, 2\pi] \times \mathbb{R}^n \).

Moreover, we assume the following:

\((H_1)\) \( h \) is bounded; i.e., for each \( i = 1, 2, 3, \ldots, n \), there exists \( K_i > 0 \) such that 
\[
|h_i(t, x)| \leq K_i
\]
for all \( (t, x) \in [0, 2\pi] \times \mathbb{R}^n \).

\((H_2)\) for each \( i = 1, 2, \ldots, n \),
\[
\frac{d}{dt} \frac{\partial F(x)}{\partial x_i} = \frac{\partial^2 F(x)}{\partial x_i^2} x_i'
\]
and there exists \( C_i > 0 \) such that
\[
\left| \frac{\partial^2 F(x)}{\partial x_i^2} \right| \geq C_i
\]
for all \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \).

The purpose of this work is to give existence and multiplicity results for periodic solutions of coupled Lienard system in \( \mathbb{R}^n \). This paper was motivated by the results in [1] and so our results in this work extend some results in [1]. To prove our results we adapt Mawhin's continuation theorem in [2], and we give appropriate region for the system's multiplicity by finding an a'priori bound.

2. A'priori Bound

To prove our assertion, we consider the following homotopy:

\((E_1)\)
\[ x''(t) + \lambda \frac{d}{dt} [VF(x(t))] + \lambda g(x) + \lambda h(t, x) = \lambda e(t). \]

Let \( \lambda \in (0, 1) \) and let \( x(t) \) be a possible solution of the problem \((E_1) (B)\). Taking \( L^2 \)-inner product by \( x'(t) \) on both sides of \((E_1)\), we have

\[ \lambda \sum_{i=1}^n \int_0^{2\pi} \frac{\partial^2 F(x(t))}{\partial x_i^2} [x_i'(t)]^2 dt + \lambda \sum_{i=1}^n \int_0^{2\pi} g_i(x(t)) x_i'(t) dt \]
\[ + \lambda \sum_{i=1}^n \int_0^{2\pi} h_i(t, x(t)) x_i'(t) dt = \lambda \sum_{i=1}^n \int_0^{2\pi} e_i(t) x_i'(t) dt. \]

By the continuity of \( \frac{\partial^2 F(x)}{\partial x_i^2} \), \((H_2)\) and the periodicity of \( x_i(t) \) in \( t \), we have

\[
\sum_{i=1}^n C_i \int_0^{2\pi} [x_i'(t)]^2 dt \leq \left( \sum_{i=1}^n \left| \int_0^{2\pi} \frac{\partial^2 F(x)}{\partial x_i^2} [x_i'(t)]^2 dt \right| \right)^{1/2} \leq \sum_{i=1}^n \sqrt{2\pi} \left( \int_0^{2\pi} [x_i'(t)]^2 dt \right)^{1/2} \leq \left( \sum_{i=1}^n \int_0^{2\pi} [x_i'(t)]^2 dt \right)^{1/2}.
\]

Hence

\[
\| x' \|_{L^2} \leq \left( \frac{1}{\min_{i=1,n} C_i} \right)^{1/2} \left( \sqrt{2\pi} \left( \sum_{i=1}^n K_i^2 \right)^{1/2} + \| e \|_{L^2} \right) = M_0.
\]

By the Sobolev inequality, we have

\[
\| x \|_{\infty} \leq \sqrt{\frac{\pi}{6}} M_0 = M_1.
\]
Suppose there exist \( a = (a_1, a_2, \ldots, a_n) \), \( b = (b_1, b_2, \ldots, b_n) \) in \( \mathbb{R}^n \) such that \( a \leq b \); if \( x(t) \) is a solution of \( (E_\alpha)(B) \) such that \( a \leq x \leq b \) and \( \|x\| = M_1 \), then

\[
\|x\| \leq \left[ \sum_{i=1}^n \left[ \max(|a_i|, |b_i|) \right]^2 \right]^{1/2} + M_1.
\]

Taking \( L^2 \)-inner product by \( x''(t) \) on both sides of \( (E_\alpha) \), we have

\[
\sum_{i=1}^n \int_0^{2\pi} [x_i''(t)]^2 dt + \lambda \sum_{i=1}^n \int_0^{2\pi} \frac{\partial^2 F(x)}{\partial x_i^2} x_i'(t)x_i''(t) dt
\]

\[
+ \lambda \sum_{i=1}^n \int_0^{2\pi} g_i(x_i(t))x_i''(t) dt + \lambda \sum_{i=1}^n \int_0^{2\pi} h_i(t,x(t))x_i''(t) dt
\]

\[
= \lambda \sum_{i=1}^n \int_0^{2\pi} \delta_i(t)x_i''(t) dt.
\]

Since \( F \) is a \( C^2 \)-function, for each \( i = 1, 2, \ldots, n \), there exists \( \alpha > 0 \) such that

\[
\left| \frac{\partial^2 F(x)}{\partial x_i^2} \right| \leq D_i,
\]

and also since \( g \) is continuous, for each \( i = 1, 2, \ldots, n \), there exists \( L_i > 0 \) such that

\[
|g_i(x_i)| \leq L_i.
\]

Hence

\[
\sum_{i=1}^n \int_0^{2\pi} [x_i''(t)]^2 dt \leq \left( \max_{1 \leq i \leq n} D_i \right) \left[ \sum_{i=1}^n \int_0^{2\pi} |x_i'(t)|^2 dt \right]^{1/2} \left[ \sum_{i=1}^n \int_0^{2\pi} |x_i''(t)|^2 dt \right]^{1/2}
\]

\[
+ \sqrt{2\pi} \left[ \sum_{i=1}^n K_i^2 \right]^{1/2} \left[ \sum_{i=1}^n \int_0^{2\pi} |x_i''(t)|^2 dt \right]^{1/2}
\]

\[
+ \left[ \sum_{i=1}^n \int_0^{2\pi} |\delta_i(t)|^2 dt \right]^{1/2} \left[ \sum_{i=1}^n \int_0^{2\pi} |x_i''(t)|^2 dt \right]^{1/2}
\]

and thus we have

\[
\|x''\|_{L^2} \leq \left( \max_{1 \leq i \leq n} D_i \right) M_0 + \sqrt{2\pi} \left[ \sum_{i=1}^n L_i^2 \right]^{1/2} + \left[ \sum_{i=1}^n K_i^2 \right]^{1/2} + \|\delta\|_{L^2} = M_2.
\]

By the Sobolev inequality

\[
\|x''\| = \sqrt{\frac{\pi}{6}} M_2
\]

for every solution of the problem \( (E_\alpha)(B) \) where \( M_2 \) depends on \( a, b, M_0 \) and \( M_1 \).

3. **OPERATOR FORMULATION**

Define

\[
L : D(L) \subseteq C^1([0, 2\pi], \mathbb{R}^n) \to L^2([0, 2\pi], \mathbb{R}^n)
\]

by

\[
(x_1(t), x_2(t), \ldots, x_n(t)) \to (x_1''(t), x_2''(t), \ldots, x_n''(t))
\]

where \( D(L) = C^2([0, 2\pi], \mathbb{R}^n) \). Then \( \text{Ker} \, L = R^2 \) and
Consider two continuous projections
\[ P : C^1([0, 2\pi], R^*) \to C^1([0, 2\pi], R^*) \]
such that
\[ \text{Im} P = \text{Ker} L \]
and
\[ Q : L^2([0, 2\pi], R^*) \to L^2([0, 2\pi], R^*) \]
defined by
\[ (Qe)(t) = \frac{1}{2\pi} \int_0^{2\pi} e(t) dt. \]
Then
\[ \text{Ker} Q = \text{Im} L, C([0, 2\pi], R^*) = \text{Ker} L \oplus \text{Im} L \]
and \( L^2([0, 2\pi], R^*) = \text{Im} L \oplus \text{Im} Q \) as a topological sum. Since
\[ \dim [L^2([0, 2\pi], R^*)/\text{Im} L] = \dim [\text{Im} Q] = \dim [\text{Ker} L] = n, \]
\( L \) is a Fredholm mapping of index zero and hence there exists an isomorphism \( J : \text{Im} Q \to \text{Ker} L \). The operator \( L \) is not bijective but the restriction of \( L \) on \( \text{Dom} L \cap \text{Ker} P \) is one-to-one and onto \( \text{Im} L \), so it has its algebraic right inverse \( K \), and as well known, it is compact. Define
\[ N : C^1([0, 2\pi], R^*) \to L^2([0, 2\pi], R^*) \]
by
\[ x(t) \to -\frac{d}{dt} [\nabla F(x(t))] - g(x(t)) - h(t, x(t)) + e(t) \]
where \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \). Then \( N \) is continuous and maps bounded sets into bounded sets. Let \( G \) be any open bounded subset of \( C^1([0, 2\pi], R^*) \), then \( QN : G \to L^2([0, 2\pi], R^*) \) is bounded and \( K \in (I - Q) : G \to L^2([0, 2\pi], R^*) \) is compact and continuous. Hence \( N \) is \( L \)-compact on \( G \). Now we see \( x \in D(L) \) is a solution to the problem \((E_1)(B)\) if and only if
\[ Lx = \lambda Nx. \]

4. MAIN RESULTS

THEOREM 4.1. Besides conditions on \( F, g, e \), and \((H_1), (H_2)\), we assume

\[(H_3) \text{ there exists } r = (r_1, r_2, \ldots, r_s), s = (s_1, s_2, \ldots, s_n), A = (A_1, A_2, \ldots, A_s) \text{ and } B = (B_1, B_2, \ldots, B_n) \text{ in } R^n \]
such that \( r < s \) and \( A \preceq B \)
\[ \frac{1}{2\pi} \int_0^{2\pi} g(r + \bar{x}(t)) dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x} + \bar{x}(t)) dt \leq A \]
and
\[ \frac{1}{2\pi} \int_0^{2\pi} g(s + \bar{x}(t)) dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x} + \bar{x}(t)) dt \geq B \]
for every \( \bar{x} \in R^s \) such that
\[ \| \bar{x} \| \leq \left( \sum_{i=1}^s [\max(\{r_i\}, |s_i|^2)]^{1/2} \right), \]
and for every \( \dot{x} \in C^1([0,2\pi),\mathbb{R}^n) \) having mean value zero, satisfying the boundary condition \( (B) \) and such that

\[
\| \dot{x} \|_{L^1} \leq \sqrt{\frac{\pi}{6}} \left( \frac{1}{\min_{i_1,\ldots,i_n} C_i} \right) \left[ \sqrt{2\pi} \left[ \sum_{i_1} K_{i_1}^{1/2} + \| \ell \|_{L^2} \right] \right],
\]

Then \((E)(B)\) has at least one solution if

\[
A < \frac{1}{2\pi} \int_0^{2\pi} e(t)\,dt < B.
\]

**PROOF.** We construct a bounded open set \( \Omega \) in \( C^1([0,2\pi),\mathbb{R}^n) \) to apply Mawhin's continuation theorem in [2]. Using a'priori estimate, we have

\[
\| x' \|_{L^1} \leq \left( \frac{1}{\min_{i_1,\ldots,i_n} C_i} \right) \left[ \sqrt{2\pi} \left[ \sum_{i_1} K_{i_1}^{1/2} + \| \ell \|_{L^2} \right] \right] = M_0,
\]

for any solution \( x(t) \) of \((E)(B), \lambda \in (0,1)\). Hence \( \| \dot{x} \|_{L^1} \leq \sqrt{\frac{\pi}{6}} M_0 = M_1 \). Define a bounded set \( \Omega^0 \) by

\[
\Omega^0 = \{ x \in C^1([0,2\pi),\mathbb{R}^n) \mid r < \| x \|_{L^1} < s, \| x \|_{L^1} < M_1 \}.
\]

Then, for any solution \( x(t) \) of \((E)(B)\) lying in \( \Omega^0 \), we have

\[
\| x \|_{L^1} \leq \left[ \sum_{i_1} \left[ \max(|r_{i_1}|,|s_{i_1}|) \right]^2 \right]^{1/2} + M_1
\]

and

\[
\| x'' \|_{L^1} \leq \left( \max_{i_1,\ldots,i_n} D_{i_1} \right) M_0 + \sqrt{2\pi} \left[ \sum_{i_1} \left( \alpha_{i_1} L_{i_1}^2 + \| \ell \|_{L^2} + \| \ell \|_{L^2} \right) \right]^{1/2} = M_2,
\]

where \( \lambda \) depends on \( r, s \) and \( M_1 \). Thus \( \| x' \|_{L^1} < \sqrt{\frac{\pi}{6}} M_2 \). Define a bounded open set \( \Omega \) by

\[
\Omega = \left\{ x \in C^1([0,2\pi),\mathbb{R}^n) \mid r < \| x \|_{L^1} < s, \| x \|_{L^1} < 2M_1, \| x' \|_{L^1} < \sqrt{\frac{2\pi}{6}} M_2 \right\}.
\]

Let \((x,\lambda) \in [D(L) \cap \partial \Omega] \times (0,1)\) and if \((x,\lambda)\) is any solution to \( Lx + \lambda Nx \), then \((x,\lambda)\) is a solution to the problem \((E)(B)\),

\[
\| \dot{x} \|_{L^1} \leq \left[ \sum_{i_1} \left[ \max(|r_{i_1}|,|s_{i_1}|) \right]^2 \right]^{1/2}, \quad \| x \|_{L^1} \leq M_1
\]

and there exists some \( i \in \{1,2,\ldots,n\} \) such that \( \dot{x}_i = r_i \) or \( s_i \). Take \( L^2\)-inner product with \( e_i = (0,0,\ldots,0,1,0,\ldots,0) \) on both sides of \((E)_i\), we have

\[
\lambda \int_0^{2\pi} g_i(x_i(t))\,dt + \lambda \int_0^{2\pi} h_i(t,x(t))\,dt = \lambda \int_0^{2\pi} e_i(t)\,dt,
\]

or

\[
\int_0^{2\pi} g_i(x_i(t))\,dt + \int_0^{2\pi} h_i(t,x(t))\,dt - \int_0^{2\pi} e_i(t)\,dt = 0
\]

if \( \dot{x}_i = r_i \), then, by assumption

\[
\int_0^{2\pi} g_i(r_i + \dot{x}_i(t))\,dt + \int_0^{2\pi} h_i(t,\dot{x}_i(t),x_1(t),\ldots,r_i,\ldots,\dot{x}_n(t),\ldots,\dot{x}_n(t))\,dt - \int_0^{2\pi} e_i(t)\,dt < 0.
\]

If \( \dot{x}_i = s_i \), then again by assumption,
Thus, for each \( \lambda \in (0, 1) \), for every solution of

\[
Lx = \lambda Nx
\]

is such that \( x \notin \partial \Omega \).

Next, we will show that \( QNx \neq 0 \) for each \( x \in \text{Ker}L \cap \partial \Omega \) and \( d_0[JQN, \Omega \cap \text{Ker}L, 0] \neq 0 \) where \( d_0 \) is the Brouwer topological degree. Since \( J: \text{Im}Q \rightarrow \text{Ker}L \) is an isomorphism and \( \dim[\text{Im}Q] = \dim[\text{Ker}L] = n \), we may take \( J \) to be the identity on \( R^n \) and hence

\[
(JQN)(x)(t) = -\frac{1}{2\pi} \int_0^{2\pi} g(x(t))dt - \frac{1}{2\pi} \int_0^{2\pi} h(t, x(t))dt + \frac{1}{2\pi} \int_0^{2\pi} e(t)dt
\]

with, for \( i = 1, 2, \ldots, n \),

\[
(JQN)_i(x)(t) = -\frac{1}{2\pi} \int_0^{2\pi} g_i(x_i(t))dt - \frac{1}{2\pi} \int_0^{2\pi} h_i(t, x_i(t))dt + \frac{1}{2\pi} \int_0^{2\pi} e_i(t)dt
\]

where \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \).

Let \( x \in \text{Ker}L \cap \partial \Omega \), then \( x = \vec{x} \) is constant in \( R^n \),

\[
\| x \| = \left[ \sum_{i=1}^n \max(|r_i|, |s_i|) \right]^{1/2}
\]

and there exists \( i \in \{1, 2, \ldots, n\} \) such that \( x_i = r_i \) or \( s_i \). In a similar manner we have \( (QN)_i(x) \neq 0 \).

Thus \( QNx \neq 0 \) for each \( x \in \text{Ker}L \cap \partial \Omega \). It is easy to see that \( P = \Omega \cap \text{Ker}L = \Pi_i \{r_i, s_i\} \). Let

\[ P_i = \{ x \in P \mid x_i = r_i \} \quad \text{and} \quad P_i' = \{ x \in P \mid x_i = s_i \} \quad \text{and} \quad x \in P, x_i \in P_i, i = 1, 2, \ldots, n \]

Then \( x = \vec{x}, x' = \vec{x}' \) are constant with

\[
\| \vec{x} \|, \quad \| \vec{x}' \| = \left[ \sum_{i=1}^n \max(|r_i|, |s_i|) \right]^{1/2}
\]

and \( x_i = r_i, x_i' = s_i \). Hence

\[
(JQN)_i(x) = -\frac{1}{2\pi} \int_0^{2\pi} g_i(r_i)dt - \frac{1}{2\pi} \int_0^{2\pi} h_i(t, x_i, \ldots, x_n)dt + \frac{1}{2\pi} \int_0^{2\pi} e_i(t)dt > 0
\]

and

\[
(JQN)_i(x') = -\frac{1}{2\pi} \int_0^{2\pi} g_i(s_i)dt - \frac{1}{2\pi} \int_0^{2\pi} h_i(t, x_i', \ldots, x_n')dt + \frac{1}{2\pi} \int_0^{2\pi} e_i(t)dt < 0
\]

Thus \( (JQN)_i(x)(JQN)_i(x') < 0 \) for \( i = 1, 2, \ldots, n \). Therefore, by the generalized intermediate value theorem, \( d_0[JQN, \Omega \cap \text{Ker}L, 0] \neq 0 \). Hence, by Mawhin's continuation theorem, the problem \( (E)(B) \) has at least one solution in \( D(L) \cap \partial \Omega \).

**Theorem 4.2.** Besides conditions on \( F, g, e, \) and \( (H_1) \) and \( (H_2) \), we assume

\( (H_4) \) there exists \( q = (q_1, q_2, \ldots, q_n), r = (r_1, r_2, \ldots, r_n), s = (s_1, s_2, \ldots, s_n), A = (A_1, A_2, \ldots, A_n) \) and \( B = (B_1, B_2, \ldots, B_n) \) in \( R^n \) such that \( q < r < s \) and \( A = B \) such that

\[
\frac{1}{2\pi} \int_0^{2\pi} g(q + \vec{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \vec{x} + \vec{x}(t))dt \geq B,
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} g(r + \vec{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \vec{x} + \vec{x}(t))dt \leq A,
\]

where \( \vec{x} = (x_1(t), x_2(t), \ldots, x_n(t)) \).
and

\[ \frac{1}{2\pi} \int_0^{2\pi} g(x + \tilde{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, x + \tilde{x}(t))dt = B \]

for every \( \tilde{x} \in \mathbb{R}^n \) such that

\[ \| \tilde{x} \| = \left( \sum_{i=1}^{n} \max(|q_i|, |r_i|, |s_i|)^2 \right)^{1/2} \]

and for every \( \tilde{x} \in C^1([0,2\pi], \mathbb{R}^n) \) having mean value zero, satisfying the boundary condition (B) such that

\[ \| \tilde{x} \| = \sqrt{\frac{1}{6} \left( \min_{x \in x^*} C_i \right)} \left[ \sqrt{2\pi} \left( \sum_{i=1}^{n} K_i^2 \right)^{1/2} + \| \tilde{e} \| \right] \]

Then \( (E)(B) \) has at least \( 2^s \) solutions if

\[ A < 1/2 \pi \int_0^{2\pi} e(t)dt < B. \]

**Proof.** We construct \( 2^s \) bounded open sets in \( C^1([0,2\pi], \mathbb{R}^n) \) to apply Mawhin’s continuation theorem in [3]. Using a’priori estimate, we have

\[ \| x \|_{L^2} \leq \left( \frac{1}{\min_{x \in x^*} C_i} \right) \left[ \sqrt{2\pi} \left( \sum_{i=1}^{n} K_i^2 \right)^{1/2} + \| \tilde{e} \| \right] = M_0 \]

for any solution \( x(t) \) of \( (E)\alpha(B), \alpha \in (0,1) \). Hence \( \| x \|_{L^2} \leq \sqrt{\frac{1}{6}} M_0 = M_1 \). Let \( I, J \) be two disjoint subsets of \( \{1, 2, ..., n\} \) such that \( I \cup J = \{1, 2, ..., n\} \) and define \( \Omega_{ij} \) by \( \Omega_{ij} = \{ x \in C^1([0,2\pi], \mathbb{R}^n) | q_i \leq \tilde{x}_i \leq r_i \}

for \( i \in I, r_j \leq \tilde{x}_j \leq s_j \) for \( j \in J, \| x \|_{L^2} = M_1 \}; \) then the number of such sets is \( 2^s \) and for any solution, \( x(t) \) of \( (E)\alpha(B) \) lying in \( \Omega_{ij} \), we have

\[ \| x \|_{L^2} \leq \left( \sum_{i \in I} \max(|q_i|, |r_i|) \right)^2 + \left( \sum_{j \in J} \max(|r_j|, |s_j|) \right)^2 \]

and

\[ \| x'' \|_{L^2} \leq \left( \max_{x \in x^*} D_i \right) M_0 + \sqrt{2\pi} \left( \sum_{i=1}^{n} L_i^2 \right)^{1/2} + \left( \sum_{i=1}^{n} K_i^2 \right)^{1/2} + \| \tilde{e} \|_{L^2} = M_2 \]

where \( L_i \) depends on \( q_i, r_i, s_i \) and \( M_2 \). Thus \( \| x \|_{L^2} \leq \sqrt{\frac{1}{3}} M_2 \). Define a bounded open set \( \Omega_{ij} \) by

\[ \Omega_{ij} = \{ x \in C^1([0,2\pi], \mathbb{R}^n) | q_i \leq \tilde{x}_i \leq r_i \}

for \( i \in I, r_j \leq \tilde{x}_j \leq s_j \}

\[ \Omega_{ij} = \{ x \in C^1([0,2\pi], \mathbb{R}^n) | q_i \leq \tilde{x}_i \leq r_i \}

for \( j \in J, \| x \|_{L^2} < 2M_1, \| x'' \|_{L^2} < \sqrt{\frac{5\pi}{3}} M_2 \).

Let \( (x, \lambda) \in [D(L) \cap \partial\Omega_{ij}] \times (0,1) \) and if \( (x, \lambda) \) is any solution to

\[ Lx = \lambda Nx, \]

then \( (x, \lambda) \) is a solution to the problem \( (E)\alpha(B) \),

\[ \| \tilde{x} \| \leq \left( \sum_{i \in I} \max(|q_i|, |r_i|) \right)^2 + \left( \sum_{j \in J} \max(|r_j|, |s_j|) \right)^2 \]

and there exists some \( i \in \{1, 2, ..., n\} \), such that \( \tilde{x}_i = q_i, r_i \) or \( s_i \). By \( (H_2) \) and assumption we can see for each \( \lambda \in (0,1) \), for every solution of \( Lx = \lambda Nx \) is such that \( x \notin \partial\Omega_{ij} \). And similarly, we can also see \( QNx \neq 0 \) for each \( x \in KerL \cap \partial\Omega_{ij} \). It is easy to see \( P = \Omega_{ij} \cap KerL = \Pi_{i \in I}[q_i, r_i] \times \Pi_{j \in J}[r_j, s_j] \). Let
and let $x \in P_i$, $x' \in P'_i$ with $i \in I \cup J$. Then, for $i \in I$, we have $x_i = q_i$, $x_i = r_i$. Hence $(JQN)(x)(JQN)(x') < 0$ for $i \in I$. For $j \in J$, we have $x_j = r_j$, $x'_j = s_j$. Thus $(JQN)(x)(JQN)(x') < 0$ for $j \in J$. Therefore, we have $d \delta [JQN, \Omega_{ij} \cap \text{Ker} L, 0] \neq 0$. Thus, by Mawhin's continuation theorem, the problem $(E_0)(B)$ has at least one solution in $D(L) \cap \overline{\Omega}_{ij}$. Thus $(E_0)(B)$ has at least $2^*$ solutions.

**Corollary 4.3.** Besides the conditions on $F$, $g$, and $e$, and $(H_1)$ and $(H_2)$, we assume

- $(H_3)$ there exists $T = (T_1, T_2, \ldots, T_n) > 0$ in $\mathbb{R}^n$ such that
  
  \[ g(T + x) - g(x) \quad \text{and} \quad h(t, T + x) - h(t, x) \]

  for all $(t, x) \in [0, 2\pi] \times \mathbb{R}^n$.

- $(H_4)$ there exists $r = (r_1, r_2, \ldots, r_n)$, $s = (s_1, s_2, \ldots, s_n)$, $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ in $\mathbb{R}^n$ such that

  \[ 0 < s - r < T, \quad r < s, \quad A < B \]

  \[ \frac{1}{2\pi} \int_0^{2\pi} g(r + x(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, T + x(t))dt \leq A, \]

  \[ \frac{1}{2\pi} \int_0^{2\pi} g(s + x(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, T + x(t))dt \geq B \]

  for every $x \in \mathbb{R}^n$ such that

  \[ \|x\| \leq \left[ \sum_{i=1}^n \left( \max(|s_i - T_i|, |r_i|, |s_i|) \right) \right]^{1/2} \]

  and for every $x \in C([0, 2\pi], \mathbb{R}^n)$ having mean value zero, satisfying the boundary condition $(B)$ and such that

  \[ \|x\|_{L^2} \leq \sqrt{\frac{\pi}{6}} \left( \frac{1}{\min_{1 \leq i \leq n} C_i} \right) \left[ \sqrt{2\pi} \left[ \sum_{i=1}^n K_i \right]^{1/2} \right] \left[ \|e\|_{L^2} \right]. \]

  Then $(E)(B)$ has at least $2^*$ solutions if

  \[ A < \frac{1}{2\pi} \int_0^{2\pi} e(t)dt < B. \]

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