CLASSICAL QUOTIENT RINGS OF GENERALIZED MATRIX RINGS

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ABSTRACT. An associative ring $R$ with identity 1 is a generalized matrix ring with idempotent set $E$ if $E$ is a finite set of orthogonal idempotents of $R$ whose sum is 1. We show that, in the presence of certain annihilator conditions, such a ring is semiprime right Goldie if and only if $eRe$ is semiprime right Goldie for all $e \in E$, and we calculate the classical right quotient ring of $R$.

KEY WORDS AND PHRASES. Generalized matrix ring, quotient ring, Goldie conditions.

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1. INTRODUCTION.

Each ring considered in this paper is associative and has an identity. Such a ring $R$ is a generalized matrix ring with idempotent set $E$ if $E$ is a finite set of orthogonal idempotents of $R$ whose sum is 1.

In this paper, we show that, in the presence of certain non-degeneracy conditions, a generalized matrix ring $R$ with idempotent set $E$ is semiprime right Goldie if and only if $eRe$ is semiprime right Goldie for all $e \in E$, and we calculate the classical right quotient ring of $R$. Kerr's example [4] of a right Goldie ring whose matrix ring is not right Goldie shows that our semiprimeness condition cannot be omitted.

Examples of generalized matrix rings include incidence algebras of directed graphs with a finite number of vertices (see [5] and [9]), structural matrix rings (see Van Wyk [13] and subsequent papers), endomorphism rings of finite direct sums of modules and Morita context rings. Sands [10] observed that if $[S,V,W,T]$ is a Morita context, then

$$\begin{bmatrix} S & V \\ W & T \end{bmatrix}$$

is a ring. These Morita context rings are precisely generalized matrix rings with idempotent sets $E$ such that $|E| = 2$, and they have been widely studied. In particular, we note Amitsur's paper [1], the survey paper [6], McConnell and Robson's treatment in ([7], 1.1 and 3.6) and
Müller’s computation of the maximal quotient ring in [8].

A generalized matrix ring \( R \) with idempotent set \( E \) is called a piecewise domain if for all \( e, f, g \in E \), \( x \in eRf \) and \( y \in fRg \), we have \( xy = 0 \) implies \( x = 0 \) or \( y = 0 \). These rings have been studied in some detail—see, for instance, [2] and [3].

We denote the prime radical of a ring \( R \) by \( p(R) \) and if \( e \) and \( f \) are idempotents of \( R \), \( e \neq f \), \( p(eRf) \) denotes the set \( \{x \in eRf : xfRe \subseteq p(Re)\} \).

**Proposition 1.1 (Sands [10]).** If \( R \) is a generalized matrix ring with idempotent set \( E \), then

\[
p(R) = \sum_{e, f \in E} p(eRf).
\]

**Proof.** If \( |E| = 1 \) there is nothing to prove and if \( |E| = 2 \) this is Theorem 1 in Sands [10]. Assume now that \( |E| = n > 2 \) and that the theorem is true for generalized matrix rings with idempotent sets of cardinality less than \( n \). Let \( e \in E \) and set \( E' = \{e, 1 - e\} \). Then \( R \) is a generalized matrix ring with idempotent set \( E' \) and so Sands’ result implies that

\[
p(R) = p(eRe) + p(eR(1 - e)) + p((1 - e)Re) + p((1 - e)R(1 - e)).
\]

Since \( (1 - e)R(1 - e) \) is a generalized matrix ring with idempotent set \( E_1 = E' \setminus \{e\} \), our induction hypothesis implies that

\[
p((1 - e)R(1 - e)) = \sum_{f, g \in E_1} p(fRg).
\]

Also, it is clear that \( p(eR(1 - e)) = \sum_{f \in E_1} p(eRf) \) and \( p((1 - e)Re) = \sum_{f \in E_1} p(fRe) \), so the result follows.

Let \( R \) be a generalized matrix ring with idempotent set \( E \). We say that the pair \( (R, E) \) satisfies the **left (respectively, right) annihilator condition** if for all \( e, f \in E \), \( 0 \neq x \in eRf \) implies that \( xfRe \neq 0 \) (respectively, \( fRe \neq 0 \)). This concept is defined in [12] where right and left are interchanged.

**Corollary 1.2. (Wauters and Jespers [12]).** The following conditions on a generalized matrix ring with idempotent set \( E \) are equivalent.

(a) \( R \) is semiprime.
(b) \( (R, E) \) satisfies the left annihilator condition and \( eRe \) is semiprime for all \( e \in E \).
(c) \( (R, E) \) satisfies the right annihilator condition and \( eRe \) is semiprime for all \( e \in E \).

2. **The Goldie Conditions.**

The right singular ideal of a ring \( S \) will be denoted by \( Z(S) \), and the right singular submodule of a right \( S \)-module \( M \) will be denoted by \( Z(M) \). So, if \( R \) is a generalized matrix ring with idempotent set \( E \) and \( e, f \in E \) with \( e \neq f \), then \( Z(eRe) \) is the right singular ideal of the ring \( eRe \) and \( Z(eRf) \) is the right singular submodule of the right \( fRf \)-module \( eRf \).

**Proposition 2.1.** Let \( R \) be a generalized matrix ring with idempotent set \( E \) and suppose that \( (R, E) \) satisfies the left annihilator condition. Then

\[
Z(R) = \sum_{e, f \in E} Z(eRf).
\]

**Proof.** Let \( e, f \in E \) and suppose that \( x \in Z(R) \). Then \( exf \in Z(R) \), so there is an essential right ideal \( I \) of \( R \) such that \( exfI = 0 \). To show that \( exf \in Z(eRf) \) it suffices to show that \( fI f \) is an essential right ideal of \( fRf \). Let \( A \) be a nonzero right ideal of \( fRf \). Because \( I \) is essential,
$I \cap AR \neq 0$. Let $0 \neq u \in I \cap AR$. Since $I \cap AR$ is a right ideal there is an idempotent $g \in E$ such that $0 \neq ug \in I \cap AR$, and $ug = fu$ because $ug \in AR \subseteq fRf$. Since $(R, E)$ satisfies the left annihilator condition, $0 \neq (fu)gRf \subseteq (I \cap AR) \cap fRf \subseteq fRf \cap A$. It follows that

$$Z(R) \subseteq \sum_{e,f \in E} Z(eRf).$$

Conversely, suppose that $e,f \in E$ and $y = ef \in Z(eRf)$. Then $yH = 0$ for some essential right ideal $H$ of $fRf$. Let $J = \{r \in R : rf \in HR\}$. Clearly, $J$ is a right ideal of $R$ and $yJ = eyfJ = (eyf)fJ \subseteq (eyf)HR = 0$, so to show that $y \in Z(R)$ it is enough to show that $J$ is essential in $R$. Let $B$ be a nonzero right ideal of $R$. If $fB = 0$, then $B \subseteq J$ and so $B \cap J \neq 0$. Now assume $fB \neq 0$. Then $fBg \neq 0$ for some $g \in E$, and so the left annihilator condition implies that $fBf \neq 0$. So we see that $fBf$ is a nonzero right ideal of $fRf$. Thus $fBf \cap H \neq 0$ and so $B \cap J \neq 0$ because $H \subseteq HR$.

**COROLLARY 2.2.** If $R$ is a generalized matrix ring with idempotent set $E$ such that $(R, E)$ satisfies the left annihilator condition, then $R$ is nonsingular if and only if $eRe$ is nonsingular for all $e \in E$.

**PROOF.** In view of the proposition, we need only show that $Z(R) \neq 0$ implies that $Z(eRe) \neq 0$ for some $e \in E$. Suppose that $0 \neq x \in Z(R)$. Then $0 \neq exf \in Z(R)$ for some $e, f \in E$. The right annihilator condition implies that $(ef)fRe \neq 0$ and so $eRe \cap Z(R) \neq 0$. It now follows from the proposition that $Z(eRe) \neq 0$.

The right uniform dimension of a ring $R$ (respectively, right $R$-module $M$) will be denoted by $d(R)$ (respectively, $d(M)$).

**PROPOSITION 2.3.** Let $R$ be a generalized matrix ring with idempotent set $E$ such that $(R, E)$ satisfies the left annihilator condition. If $d(R) < \infty$ then $d(eRe) < \infty$ for all $e \in E$. Moreover, if $R$ is semiprime and $d(eRe) < \infty$ for all $e \in E$, then $d(eRf) < \infty$ for all $e, f \in E$ and hence $d(R) < \infty$.

**PROOF.** Assume that $d(R) < \infty, e \in E$ and $\sum A_i$ is a direct sum of nonzero right ideals of $eRe$. To prove that $d(eRe) < \infty$ it is enough to show that $\sum A_iR$ is direct, and to accomplish this we need only show that $\sum A_iRf$ is direct for each $f \in E$. Suppose that $f \in E$ and $b_i \in A_iRf$ are such that $\sum b_i = 0$. Since $b_iRf \subseteq A_i$, and $\sum A_i$ is direct, $b_iRf = 0$ for all $i$. Thus the left annihilator condition implies that $b_i = 0$ for all $i$ and hence $\sum A_iRf$ is direct.

Now assume that $d(eRe) < \infty$ for all $e \in E$ and suppose that $\sum N_i$ is a direct sum of nonzero $fRf$-submodules of $eRf$. Since $0 \neq N_i \subseteq eRf$ the left annihilator condition implies that $N_i, fRe \neq 0$ and each $N_i, fRe$ is a right ideal of $eRe$. Let $K = N, fRe \cap \sum \{N_i : i \neq j\}$. Then $KeRf \subseteq N_j \cap \sum \{N_i : i \neq j\}$ and so $KeRf = 0$. Since $K^2 \subseteq KeRf$, $K^2 = 0$ and so $K = 0$ because $eRe$ is semiprime by Corollary 2. Thus $\sum N_i fRe$ is direct and so $d(eRe) \leq d(eRf)$. It follows that $R$ has finite right uniform dimension as a right $(\sum eRe)$-module and so certainly $d(R) < \infty$.

**THEOREM 2.4.** Let $R$ be a generalized matrix ring with idempotent set $E$. If $R$ is semiprime right Goldie, then so too are the rings $eRe, e \in E$. Conversely, if $(R, E)$ satisfies the left annihilator condition and $eRe$ is semiprime right Goldie for all $e \in E$, then $R$ is semiprime right Goldie.

**3. THE QUOTIENT RING.**

Let $S$ and $T$ be rings and let $M$ be an $S - T$-bimodule. We say that $M$ satisfies the right
PROPOSITION 3.1. If $R$ is a semiprime right Goldie generalized matrix ring with idempotent set $E$, then $eRF$ satisfies the right bimodule Ore condition for all $e, f \in E$.

PROOF. Let $m \in eRF$ and suppose that $c$ is regular in $eRe$. Define $\theta(eRF) = eRF$ by $\theta(x) = cx$ for all $x \in eRF$. Clearly $\theta$ is an $eRF$-module homomorphism and we now check that $\theta$ is a monomorphism. Suppose that $x \in eRF$ and $cx = 0$. Then $eRFx = 0$ which implies that $xfRe = 0$ because $c$ is regular. But then the left annihilator condition implies that $x = 0$ as required.

From Theorem 6 and Proposition 5 we know that $eRF$ has finite right uniform dimension as a right $fRF$-module. Since $eRF$ and $ceRF$ are isomorphic $fRF$-modules, $d(ceRF) = d(eRF)$ and so $ceRF$ is an essential $fRF$ submodule of $eRF$. Hence $\{y \in fRF: my \in ceRF\}$ is an essential right ideal of $fRF$ which, since $fRF$ is semiprime right Goldie, must contain the required regular element $c$.

If $S$ is semiprime right Goldie, $Q(S)$ denotes the classical right quotient ring of $S$ and if $M$ is a right $S$-module, $Q(M) = M \otimes_S Q(S)$. Using the right common denominator property of Ore sets we see that every element of $Q(M)$ is of the form $m \otimes c^{-1}$ where $m \in M$ and $c$ is regular in $S$. In what follows we shall write $nc^{-1}$ instead of $m \otimes c^{-1}$.

THEOREM 3.2. If $R$ is a semiprime right Goldie generalized matrix ring with idempotent set $E$, then

$$Q(R) = \sum_{e, f \in E} Q(eRF).$$

PROOF. For each $e \in E, eRe$ embeds in $Q(eRe)$ and we now check that for $e, f \in E, e \neq f, eRF$ embeds in $Q(eRF)$. Suppose that $c$ is regular in $fRF$, $x \in eRF$ and $xc = 0$. Then $Rexc = 0$ and so $Rex = 0$ because $c$ is regular in $fRF$. Thus $eRFxc = 0$ and hence $0 = exf = x$ since $R$ is semiprime. This shows that $eRF$ is a torsion free $fRF$-module and so $eRF$ embeds in $Q(eRF)$.

Let $e, f, g \in E$, $x \in eRF$, $y \in fRG$ and suppose that $c$ is regular in $fRF$ and $d$ is regular in $gRG$. Define $(xc^{-1})(yd^{-1}) = xyc_1d_1$ where $y_1$ and $c_1$ are obtained from the right bimodule Ore condition: $yc_1 = c_1y$. It is straightforward to check that this multiplication is well-defined and that as a result $Q = \sum_{e \in E} Q(eRF)$ becomes a generalized matrix ring with idempotent set $E$.

We now show that $Q$ is semiprime. It follows from Theorem 6 that $eRe$ is semiprime right Goldie for all $e \in E$ and hence $Q(eRe)$ is semiprime for all $e \in E$. In view of Corollary 1.2, it suffices to show that $(Q, E)$ satisfies the right annihilator condition. Let $yd^{-1} \in Q(fRG)$ be such that $Q(eRF)yd^{-1} = 0$. Then $(eRF)(yd^{-1}) = 0$ and so $(eRF)y = 0$. From Corollary 1.2 we see that $(Q, E)$ satisfies the right annihilator condition and so $y = 0$. Thus $(Q, E)$ satisfies the right annihilator condition and hence $Q$ is semiprime.

Let $e, f \in E, e \neq f$. From Proposition 2.3 we see that $eRF$ has finite uniform dimension as a right $fRF$-module and so $Q(eRF)$ has finite uniform dimension as a right $Q(fRF)$-module. Since $Q(fRF)$ is semisimple Artinian it follows that $Q(eRF)$ is an Artinian $Q(fRF)$-module, and hence $Q$ is right Artinian by an argument similar to ([7], 1.1.7). Since we have already seen that $Q$ is semiprime, $Q$ is a semisimple Artinian ring.

To complete the proof, we need only show that $R$ is a right order in $Q$. Let $x \in Q, x = \sum_{e, f \in E} z(e, f)$ where $z(e, f) \in Q(eRF)$ for all $e, f \in E$. Using the right common denominator
bimodule Ore condition if for each $m \in M$ and each regular element $c \in S$ there is an $m_1 \in M$ and a regular $c_1 \in T$ such that $mc_1 = cm_1$.

**Proposition 3.1.** If $R$ is a semiprime right Goldie generalized matrix ring with idempotent set $E$, then $eRF$ satisfies the right bimodule Ore condition for all $e,f \in E$.

**Proof.** Let $m \in eRF$ and suppose that $c$ is regular in $eRe$. Define $\theta: eRF \rightarrow ceRF$ by $\theta(x) = cx$ for all $x \in eRF$. Clearly $\theta$ is an $fRF$-module homomorphism and we now check that $\theta$ is a monomorphism. Suppose that $x \in eRF$ and $cx = 0$. Then $cxRe = 0$ which implies that $xRF = 0$ because $c$ is regular. But then the left annihilator condition implies that $x = 0$ as required.

From Theorem 6 and Proposition 5 we know that $eRF$ has finite right uniform dimension as a right $fRF$-module. Since $eRF$ and $ceRF$ are isomorphic $fRF$-modules, $d(ceRF) = d(eRF)$ and so $ceRF$ is an essential $fRF$ submodule of $eRF$. Hence $\{y \in fRF: my \in ceRF\}$ is an essential right ideal of $fRF$ which, since $fRF$ is semiprime right Goldie, must contain the required regular element $c_1$.

If $S$ is semiprime right Goldie, $Q(S)$ denotes the classical right quotient ring of $S$ and if $M$ is a right $S$-module, $Q(M) = M \otimes_S Q(S)$. Using the right common denominator property of Ore sets we see that every element of $Q(M)$ is of the form $m \otimes c^{-1}$ where $m \in M$ and $c$ is regular in $S$. In what follows we shall write $m \otimes c^{-1}$ instead of $mc^{-1}$.

**Theorem 3.2.** If $R$ is a semiprime right Goldie generalized matrix ring with idempotent set $E$, then

$$Q(R) = \sum_{e,f \in E} Q(eRF).$$

**Proof.** For each $e \in E, eRe$ embeds in $Q(eRe)$ and we now check that for $e,f \in E, e \neq f, eRF$ embeds in $Q(eRF)$. Suppose that $c$ is regular in $fRF$, $x \in eRF$ and $xc = 0$. Then $fRxc = 0$ and so $fRex = 0$ because $c$ is regular in $fRF$. Thus $exfRe = 0$ and hence $0 = exf = x$ since $R$ is semiprime. This shows that $eRF$ is a torsion free $fRF$-module and so $eRF$ embeds in $Q(eRF)$.

Let $e,f,g \in E$, $x \in eRF$, $y \in fRG$ and suppose that $c$ is regular in $fRF$ and $d$ is regular in $gRG$. Define $(x^{-1})(y^{-1}) = y_1c_1^{-1}d^{-1}$ where $y_1$ and $c_1$ are obtained from the right bimodule Ore condition: $yc_1 = cy_1$. It is straightforward to check that this multiplication is well-defined and that as a result $Q = \sum_{e,f \in E} Q(eRF)$ becomes a generalized matrix ring with idempotent set $E$.

We now show that $Q$ is semiprime. It follows from Theorem 6 that $eRe$ is semiprime right Goldie for all $e \in E$ and hence $Q(eRe)$ is semiprime for all $e \in E$. In view of Corollary 1.2, it suffices to show that $(Q,E)$ satisfies the right annihilator condition. Let $yd^{-1} \in Q(fRF)$ be such that $Q(eRF)y = 0$. Then $(eRF)(yd^{-1}) = 0$ and so $(eRF)y = 0$. From Corollary 1.2 we see that $(R,E)$ satisfies the right annihilator condition and so $y = 0$. Thus $(Q,E)$ satisfies the right annihilator condition and hence $Q$ is semiprime.

Let $e,f \in E, e \neq f$. From Proposition 2.3 we see that $eRF$ has finite uniform dimension as a right $fRF$-module and so $Q(eRF)$ has finite uniform dimension as a right $Q(fRF)$-module. Since $Q(fRF)$ is semisimple Artinian it follows that $Q(eRF)$ is an Artinian $Q(fRF)$-module, and hence $Q$ is right Artinian by an argument similar to ([7], 1.1.7). Since we have already seen that $Q$ is semiprime, $Q$ is a semisimple Artinian ring.

To complete the proof, we need only show that $R$ is a right order in $Q$. Let $x \in Q$, $x = \sum_{e,f \in E} x(e,f)$ where $x(e,f) \in Q(eRF)$ for all $e,f \in E$. Using the right common denominator
property we can find, for each \( f \in E \), an \( a_f \in fRf \) and elements \( y(e, f) \in eRf \) such that for all \( e \in E \), \( x(e, f) = y(e, f)a_f^{-1} \). Let \( y = \sum_{e, f \in E} y(e, f) \) and \( z = \sum_{f \in E} a_f \). Then \( x = yz^{-1} \) and so \( R \) is a right order in \( Q \). \( \square \)

Let \( R \) be a semiprime right Goldie ring with idempotent \( e \). Clearly, \( R \) is a generalized matrix ring with idempotent set \( E = \{ e, 1 - e \} \) and so it follows from Theorem 2.4 that \( eRe \) is semiprime right Goldie. This result seems to have been well-known for some time. Also, it follows from Theorem 3.2 that the classical right quotient ring of \( eRe \) is \( eQe \) where \( Q \) is the classical right quotient ring of \( R \). This result is due to Small [11].

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