NOTE ON HÖLDER INEQUALITIES

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ABSTRACT. In this note, we show that if $m, n$ are positive integers and $x_{ij} \geq 0$, for $i = 1, \ldots, n$, for $j = 1, \ldots, m$, then

$$\left( \sum_{i=1}^{n} x_{i1} \cdots x_{im} \right)^{m} \leq \left( \sum_{i=1}^{n} x_{il}^{m} \right) \cdots \left( \sum_{i=1}^{n} x_{im}^{m} \right)$$

with equality, in case $(x_{11}, \ldots, x_{n1}) \neq 0$ if and only if each vector $(x_{1j}, \ldots, x_{nj})$, $j = 1, \ldots, m$, is a scalar multiple of $(x_{11}, \ldots, x_{n1})$. The proof is a straight-forward application of Hölder inequalities. Conversely, we show that Hölder inequalities can be derived from the above result.

KEY WORDS AND PHRASES. The Hölder Inequalities.

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1. MAIN RESULTS.

LEMMA 1. If $m, n$ are positive integers and $x_{ij} \geq 0$, for $i = 1, \ldots, n$, for $j = 1, \ldots, n$, then

$$\left( \sum_{i=1}^{n} x_{i1} \cdots x_{im} \right)^{m} \leq \left( \sum_{i=1}^{n} x_{il}^{m} \right) \cdots \left( \sum_{i=1}^{n} x_{im}^{m} \right)$$

with equality, in case $(x_{11}, \ldots, x_{n1}) \neq 0$ if and only if each vector $(x_{1j}, \ldots, x_{nj})$, $j = 1, \ldots, m$, is a scalar multiple of $(x_{11}, \ldots, x_{n1})$.

PROOF. Use induction on $m$. When $m = 1$, the above inequalities are trivial. Suppose that the above inequalities hold with $m - 1$. Then it follows that

$$\left( \sum_{i=1}^{n} x_{i1} \cdots x_{im} \right)^{m} \leq \left( \sum_{i=1}^{n} (x_{i1} \cdots x_{im-1})^{m-1} \right)^{m} \cdot \left( \sum_{i=1}^{n} x_{im} \right)^{1 \cdot m}$$

(by Hölder Inequalities)

$$= \left\{ \sum_{i=1}^{n} x_{im}^{m} \cdot \sum_{i=1}^{n} x_{im-1}^{m-1} \right\}^{m} \cdot \left\{ \sum_{i=1}^{n} x_{im} \right\}^{1 \cdot m}$$

(by Induction Hypothesis)

$$= \left\{ \sum_{i=1}^{n} x_{i1}^{m} \cdots x_{im-1}^{m} \cdot x_{im} \right\}^{1 \cdot m}$$

Therefore the proof is complete.
Note that the above inequalities have been deduced using Hölder Inequalities. We can also deduce Hölder Inequalities by using the above inequalities.

**THEOREM 1.** Given \( p_1, \cdots, p_n \in \mathbb{R} \) with \( p_k > 1 \), for each \( k = 1, \cdots, n \) and \( \sum_{k=1}^{n} \frac{1}{p_k} = 1 \) and given \( a_1, \cdots, a_n > 0 \), we have the following inequality

\[
a_1 \cdots a_n \leq \sum_{k=1}^{n} \frac{a_k^{p_k}}{p_k}.
\]

**PROOF.** First we prove this theorem when all \( p_k \)'s are rational. Write \( p_k = \frac{c_k}{b_k} \) for some \( b_k, c_k \in \mathbb{N} \) for \( 1 \leq k \leq n \). Let \( m = 2 \cdot \text{lcm}(c_1, \cdots, c_n) \). Let \( q_k = \frac{m}{b_k} \) for \( 1 \leq k \leq n \). It is clear that \( q_k \geq 2 \) for \( 1 \leq k \leq n \). Let \( x_k = a_k^{q_k} \) for \( 1 \leq k \leq n \). Let \( S : \mathbb{R}^{n} \rightarrow \mathbb{R}^m \) be the mapping defined by

\[
S(y_1, y_2, \cdots, y_m) = (y_m, y_1, y_2, \cdots, y_{m-1})
\]

for \( (y_1, y_2, \cdots, y_m) \in \mathbb{R}^m \). Define \( m \) vectors \( Z_1, \cdots, Z_m \) by

\[
Z_1 = \left( \frac{q_1 - \text{times}}{x_1, \cdots, x_1}, \frac{q_2 - \text{times}}{x_2, \cdots, x_2}, \cdots, \frac{q_m - \text{times}}{x_m, \cdots, x_m} \right)
\]

and \( Z_i = S(Z_{i-1}) \) for \( 2 \leq i \leq m \). Applying the Lemma 1 to the \( m \) vectors \( Z_1, \cdots, Z_m \), we have

\[
m \cdot x_1^{q_1} \cdots x_n^{q_n} \leq q_1 \cdot x_1^{m} + \cdots + q_n \cdot x_n^{m}
\]

and equality holds if and only if \( x_1 = x_2 \) for \( 2 \leq k \leq n \).

By substituting \( x_k^{m} = a_k^{p_k}(1 \leq k \leq n) \) into both sides in (1.1), we have

\[
a_1 \cdots a_n \leq \sum_{k=1}^{n} \frac{a_k^{p_k}}{p_k},
\]

and equality holds if and only if \( a_1^{p_1} = a_k^{p_k} \) for \( 2 \leq k \leq n \). Now, let us show the theorem when all \( p_k \)'s are real. We can choose \( n \) sequences of rational numbers \( \{r_{i,j}\}, \cdots, \{r_{n,j}\} \) satisfying \( r_{k,j} > 1 \) for \( 1 \leq k \leq n \), all \( j \in \mathbb{N} \) and \( \sum_{k=1}^{n} \frac{1}{k} = 1 \) for each \( j \in \mathbb{N} \) and \( r_{k,j} \rightarrow p_k \) as \( j \rightarrow \infty \), for \( 1 \leq k \leq n \). By the above argument, for each \( j \in \mathbb{N} \), we have

\[
a_1 \cdots a_n \leq \sum_{i=1}^{n} \frac{a_i^{p_{k_j}}}{r_{k_j}}
\]

Taking the limit as \( j \rightarrow \infty \), the result follows.

Hölder Inequalities follow from Theorem 1 in the usual way, that can be found in most textbooks. From Lemma 1 and Theorem 1, we know that the following form of inequalities is essential for the Hölder inequalities: If \( n \) is a positive integer and \( x_{ij} \geq 0 \), for \( i = 1, \cdots, n \), for \( j = 1, \cdots, n \), then

\[
\left( \sum_{i=1}^{n} x_{i1} \cdots x_{in} \right)^{n} \leq \left( \sum_{i=1}^{n} x_{i1} \right)^{n} \cdots \left( \sum_{i=1}^{n} x_{in} \right)^{n}.
\]

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