CHARACTERIZATION OF FUZZY NEIGHBORHOOD
COMMUTATIVE DIVISION RINGS II

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ABSTRACT. In [4] we produced a characterization of fuzzy neighborhood commutative division rings; here we present another characterization of it in a sense that we minimize the conditions so that a fuzzy neighborhood system is compatible with the commutative division ring structure. As an additional result, we show that Chadwick [5] relatively compact fuzzy set is bounded in a fuzzy neighborhood commutative division ring.

KEY WORDS AND PHRASES. Fuzzy neighborhood system, fuzzy neighborhood ring, fuzzy neighborhood commutative division ring, relatively compact fuzzy set, bounded fuzzy set.

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1. PRELIMINARIES.

Just like our previous work, we consider here the fuzzy neighborhood topology \( t(\Sigma) \) on \( D \), the one generated by the well-known fuzzy neighborhood system \( \Sigma \) of R. Lowen [8]. The pair \((D, t(\Sigma))\) is termed as a fuzzy neighborhood space. The triplet \((D, +, \cdot)\) (or \( D \) alone) is considered either a ring, division ring or commutative division ring (whichever we require). \( D^* = D \setminus \{0\} \) denotes the multiplicative group of nonzero elements of the commutative division ring \( D \) and \( D^+ \) is the additive group of \( D \).

As usual \( I_\mathbb{C} = [0,1], I = [0,1] \) the unit interval, and \( 2(D) \) denotes finite subsets of \( D \). If \( \mu \) is a fuzzy set of \( D \) then \( \mu \) is given by

\[
\mu(x) = \mu(x^{-1}) \forall x \in D^* ,
\]

while for \( x \in D \),

\[
x \oplus \mu(y) = 1_x \oplus \mu(y) = \mu(y - x) \forall y \in D ,
\]

where \( 1_x \) denotes the characteristic function of the singleton set \( \{x\} \).

For any \( \mu, \nu, \theta \in 2(D) \) and \( x \in D^* \),

\[
x \odot \mu, \nu \oplus \theta \text{ and } \nu \odot \theta \text{ are defined by:}
\]

\[
x \odot \mu(y) = 1_x \odot \mu(y) = \mu(x^{-1}y) \\
\nu \oplus \theta(y) = \sup_{s + t = \nu} \nu(s) \land \theta(t)
\]

and

\[
\nu \odot \theta(y) = \sup_{st = \nu} \nu(s) \land \theta(t)
\]

\( \forall y \in D \).
We define \( \mu/\nu \) as
\[
\mu/\nu = \mu \otimes \nu^-
\]
and thus \( 1/(1 \oplus \nu) \) is written as:
\[
1/(1 \oplus \nu)(x) = (1 \oplus \nu)^-(x) = (1 \oplus \nu)(x^{-1}) \Rightarrow \]
\[
1/(1 \oplus \nu)(x) = \sup_{(t+s)^{-1} = x} \nu(s)
\]
\( \forall x \in D^* \).

We call \( \mu \) symmetric if and only if
\[
\mu = \sim \mu, \text{ where } \sim \mu(x) = \mu(-x) \forall x \in D.
\]
The constant fuzzy set of \( D \) with value \( \delta \in I \) is given by the symbol \( \delta \text{ID} \). The saturation operation [8] is defined on a prefilter base \( \mathcal{F} \subset I^D \) by \( \mathcal{F} = \{ v \in I^D : \forall \delta \in [\delta, \delta] \in \mathcal{F} \exists v_6 - \delta \leq v \} \).

**PROPOSITION 1.1** [8]. Let \( (D, t(\Sigma)) \) and \( (D, t(\Sigma')) \) be fuzzy neighborhood spaces and \( f : D \rightarrow D' \), then \( f \) is continuous at \( x \in D \in \mathcal{F} \) if and only if \( f(\delta) \in \mathcal{F} \).

**DEFINITION 1.2** [3]. Let \( (D, +, \cdot, t(\Sigma)) \) be a ring and \( \Sigma \) a fuzzy neighborhood system on \( D \). Then the quadruple \( (D, +, \cdot, t(\Sigma)) \) is said to be a fuzzy neighborhood ring if and only if the following are fulfilled:

1. \( (D, +, \cdot, t(\Sigma)) \) is a fuzzy neighborhood group [1].
2. The mapping \( \mathcal{M} : (D \times D, t(\Sigma) \times t(\Sigma)) \rightarrow (D, t(\Sigma)), (x, y) \mapsto xy \) is continuous.

**PROPOSITION 1.3** [3]. If \( (D, +, \cdot, t(\Sigma)) \) is a fuzzy neighborhood ring and \( x \in D \), then
(a) The left homothety \( L_x : (D, t(\Sigma)) \rightarrow (D, t(\Sigma)), y \mapsto xy \) (resp. right homothety \( R_x : (D, t(\Sigma)) \rightarrow (D, t(\Sigma)), y \mapsto yx \)) is continuous. If \( x \) is a unit element of \( D \) then each homothety is a homeomorphism.
(b) The translation \( T_x : (D, t(\Sigma)) \rightarrow (D, t(\Sigma)), y \mapsto y + x \) and the inversion \( k : (D, t(\Sigma)) \rightarrow (D, t(\Sigma)), x \mapsto -x \) are homeomorphisms.
(c) \( \nu \in \Sigma(0) \Leftrightarrow x + \nu \in \Sigma(x) \), i.e., \( T_x(\nu) \in \Sigma(x) \).
(d) \( \nu \in \Sigma(x) \Leftrightarrow -x + \nu \in \Sigma(0) \), i.e., \( T_x(\nu) \in \Sigma(x) \).

**DEFINITION 1.4** [2]. Let \( (D, +, \cdot) \) be a division ring, and \( \Sigma \) a fuzzy neighborhood system on \( D \). Then the quadruple \( (D, +, \cdot, t(\Sigma)) \) is said to be a fuzzy neighborhood division ring if and only if the following are true:

1. \( (D, +, \cdot, t(\Sigma)) \) is a fuzzy neighborhood ring.
2. The mapping \( r : (D^*, t(\Sigma^*)) \rightarrow (D^*, t(\Sigma^*_D)), x \mapsto x^{-1} \) is continuous where \( \Sigma^*_D \) is the fuzzy neighborhood system on \( D^* \) induced by \( D \).

A commutative division ring structure and a fuzzy neighborhood \( \Sigma \) on \( D \) are said to be compatible if the conditions \( (\text{FNDR1}) \) and \( (\text{FNDR2}) \) are satisfied.

**THEOREM 1.5** [3]. Let \( (D, +, \cdot) \) be a ring and \( \Sigma \) a fuzzy neighborhood system on \( D \). Then the quadruple \( (D, +, \cdot, t(\Sigma)) \) is a fuzzy neighborhood ring if and only if the following conditions are satisfied:

1. \( \forall x \in D : \Sigma(x) = \{ T_x(\nu) : \nu \in \Sigma(0) \} \).
2. \( \forall x \in D, \forall \mu \in \Sigma(0), \forall \nu \in \Sigma(0) \exists x_0 \circ \nu \leq \mu + \delta, \) and \( \nu \circ x_0 \leq \mu + \delta \), i.e., the mapping \( y \mapsto x_0 y \) and \( y \mapsto y x_0 \) are continuous at \( 0 \).
3. \( \forall \mu \in \Sigma(0), \forall \nu \in \Sigma(0) \exists \nu \circ \nu \leq \mu + \delta \), i.e., the mapping \( (x, y) \mapsto x + y \) is continuous at \( (0, 0) \).
4. \( \forall \mu \in \Sigma(0), \forall \nu \in \Sigma(0) \exists \nu \leq - \mu + \delta \), i.e., the mapping \( x \mapsto -x \) is continuous at \( 0 \).
5. \( \forall \mu \in \Sigma(0), \forall \nu \in \Sigma(0) \exists \nu \circ \nu \leq \mu + \delta \), i.e., the mapping \( (x, y) \mapsto xy \) is continuous at
THEOREM 1.6 [4]. Let \((D, +, \cdot, t(\Sigma))\) be a commutative division ring and \((D, +, \cdot, t(\Sigma))\) a fuzzy neighborhood ring. Then the quadruple \((D, +, \cdot, t(\Sigma))\) is a fuzzy neighborhood commutative division ring if and only if the following are fulfilled:

(i) \(\forall x \in D: \Sigma(x) = \{T_x(\nu): \nu \in \Sigma(0)\}\).
(ii) \(\forall \mu \in \Sigma(0), \forall x \in D, \forall \delta \in I_o, \forall \nu \in \Sigma(0) \exists x \odot \nu \leq \mu + \frac{\delta}{\delta} \); i.e., \(y \mapsto xy\) is continuous at 0.
(iii) \(\forall \mu \in \Sigma(0), \forall \delta \in I_o, \forall \nu \in \Sigma(0) \exists x \odot \nu \leq \mu + \frac{\delta}{\delta} \); i.e., \((x, y) \mapsto x + y\) is continuous at \((0, 0)\).
(iv) \(\forall \mu \in \Sigma(0), \forall \delta \in I_o, \forall \nu \in \Sigma(0) \exists x \odot \nu \leq \mu + \frac{\delta}{\delta} \); i.e., \((x, y) \mapsto xy\) is continuous at \((0, 0)\).
(v) \(\forall \mu \in \Sigma(0), \forall \delta \in I_o, \forall \nu \in \Sigma(0) \exists (1 \odot \nu)^{\sim} \leq (1 + \mu) + \frac{\delta}{\delta} \); i.e., the inversion \(x \mapsto x^{-1}(x \neq 0)\) is continuous at 1.

PROPOSITION 1.7 [4]. Let \((D, +, \cdot\) be a fuzzy neighborhood commutative division ring. If conditions (i)-(v) of Theorem 1.6 are satisfied, then the following inequality holds good:

\[\forall \mu \in \Sigma(0), \forall \delta \in I_o, \forall \nu \in \Sigma(0) \exists (1 \odot \nu)^{\sim} \leq (1 + \mu) + \frac{\delta}{\delta}.\]

Recently the notion of relatively \(f\)-compact fuzzy sets was introduced by Chadwick and studied in relation to fuzzy neighborhood spaces. We quote the following results from [5]. For a detailed account of compact fuzzy neighborhood spaces we refer to [9].

THEOREM 1.8 [5]. Let \((D, t(\Sigma))\) be a fuzzy neighborhood spaces, \(\mu \in I^D\). Then the following are equivalent:

(a) \(\mu\) is relatively \(f\)-compact;
(b) for each family \((\nu_x)_{x \in D} \subseteq \Pi_{x \in D} \Sigma(x)\) and each \(\delta > 0\) there is \(F \in \Sigma(0)\) such that

\[\sup_{x \in F} \nu_x \geq \mu - \frac{\delta}{\delta} .\]

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We shall now present an equivalent form of condition (v) of Theorem 1.6, namely

\[(v') \forall \mu \in \Sigma(0) \forall \delta \in I_o \forall \nu \in \Sigma(0) \exists (1 \odot \nu)^{\sim} \leq (1 + \mu) + \frac{\delta}{\delta} , \quad (2.1)\]

or

\[\nu \circ (1 \odot \nu)^{\sim} \leq \mu + \frac{\delta}{\delta} .\]

PROPOSITION 2.1. Referring to Theorem 1.6, we have conditions (i)-(v), are equivalent to

(i)-(iv) and \((v')\).

PROOF. We need prove only the converse part, i.e., \(\forall \mu \in \Sigma(0), \forall \delta \in I_o \forall \nu \in \Sigma(0) \exists (1 \odot \nu)^{\sim} \leq (1 + \mu) + \delta.\) Suppose conditions (i)-(iv) and \((v')\) hold, and \(\mu \in \Sigma(0)\) and \(\delta \in I_o\); choose \(\nu = \sim \nu\) symmetric \(\exists \nu \odot (1 \odot \nu)^{\sim} \leq \mu + \frac{\delta}{\delta}\). Let \(x \in D^*\), then

\[(1 \odot \nu)^{\sim} (x) = (1 \odot \nu)(x^{-1}) = \sup_{1 + x = z} \nu(x) = \sup_{1 - \frac{z}{1 + x} = z} \nu(x) = \sup_{1 - \frac{x}{1 + x} = z - 1} \nu \circ (1 \odot \nu)^{\sim} = \sup_{x(1 + x) = z - 1} \nu = \sup_{x(1 + x) = z - 1} \nu(\neg x) = \nu(\neg z)\]
\[ \begin{align*}
&\leq \sup_{x(1 + x)^{-1} = z} \nu(x) \wedge (1 + \nu)^\sim (1 + x)^{-1} \\
&= \sup_{x(1 + x)^{-1} = z} \nu(x) \wedge (1 + \nu)^\sim (1 + x)^{-1} \\
&= \nu \odot (1 + \nu)^\sim (z - 1) \\
&= 1 + (\nu \odot (1 + \nu)^\sim)(z) \\
&\leq (1 + \mu)(z) + \delta \text{ (by } \nu')
\end{align*} \]

i.e., \((1 + \nu)^\sim \leq (1 + \mu) + \delta\). \hfill \square

**THEOREM 2.2.** Let \((D, +, \cdot)\) be a commutative division ring and \(\Sigma\) a fuzzy neighborhood system on \(D\). Then the following conditions are equivalent:

1° \((D, +, \cdot, t(\Sigma))\) is a fuzzy neighborhood commutative division ring.

2° \((D, +, \cdot, t(\Sigma))\) is a fuzzy neighborhood group, and division

\[ d: D \times D^* \to D, (x, y) \mapsto x/y \]

is continuous.

3° \((D, +, \cdot, t(\Sigma))\) and \((D^*, \cdot, t(\Sigma_{|D^*}))\) are fuzzy neighborhood groups.

**PROOF.** 1°⇒2°. Let \((D, +, \cdot, t(\Sigma))\) be a fuzzy neighborhood commutative division ring. Then by definition it is an additive group and therefore a fuzzy neighborhood group. We show division

\[ d: D \times D^* \to D, (x, y) \mapsto x/y \]

is continuous.

But from the following scheme we have:

\[ D \times D^* \to D \times D \to D, \]

\[ (x, y) \mapsto (x, y^{-1}) \mapsto xy^{-1}, \]

i.e., \(d: D \times D^* \to D, (x, y) \mapsto x/y\) is continuous.

2°⇒3°. If the division \(d\) is continuous on \(D \times D^*\), then certainly the restriction to \(D^* \times D^*\) is continuous, i.e., \((D^*, \cdot, t(\Sigma_{|D^*}))\) is a fuzzy neighborhood group.

3°⇒1°. We need to show that \(m: D \times D \to D, (x, y) \mapsto xy\) is continuous. But this follows from Theorem 3.3 [4].

**THEOREM 2.3.** Let \((D, +, \cdot)\) be a commutative division ring with characteristic \(\text{Char}(D) \neq 2\). Then a fuzzy neighborhood group on the commutative division ring \(D\) with respect to which the inversion is continuous is a fuzzy neighborhood commutative division ring.

**PROOF.** The continuity of multiplication follows from the equality:

\[ xy = [(x + y - 2)^{-1} - (x + y + 2)^{-1}]^{-1} - [(x - y - 2)^{-1} - (x - y + 2)^{-1}]^{-1}. \]

\[ \square \]

**THEOREM 2.4.** Let \((D, +, \cdot)\) be a commutative division ring and \(\Sigma\) a fuzzy neighborhood system on \(D\) such that

(i) multiplication, \(m: D \times D \to D, (x, y) \mapsto xy\),

(ii) inversion, \(r: D^* \to D^*, x \mapsto x^{-1}\),

(iii) addition of 1, \(p: D \to D, x \mapsto 1 + x\), are continuous, then \((D, +, \cdot, t(\Sigma))\) is a fuzzy neighborhood commutative division ring.
**Proof.** Negation \( x \mapsto -x = (-1)x \) is continuous then \( x \mapsto x - 1 \) is the composite:

\[
x \mapsto -x \mapsto -x + 1 \mapsto -(-x + 1)
\]

of continuous functions and therefore, continuous.

It remains to show that the addition is continuous. To this end, we show that the addition is continuous at \((0,0)\). In order to do so, we use the following identity:

\[
x + y = [1 + y(1 + x)^{-1}](1 + x) - 1
\]

Let \( \mu \in \Sigma(0) \) and \( \delta > 0 \). Choose \( \nu \in \Sigma(0) \) and \( \theta \in \Sigma(1) \).

By continuity of \( p: x \mapsto 1 + x \), we get

\[
\theta \circ 1 \leq \mu + \delta/7 \quad (2.2)
\]

Since multiplication \( m: D \times D \rightarrow D, (x, y) \mapsto xy \) is continuous at \((1,1)\), we have a \( \theta \in \Sigma(1) \) such that

\[
\theta \circ \theta \leq \theta + \delta/7 \quad (2.3)
\]

Again applying the continuity of \( p \), we can find \( \nu \in \Sigma(0) \) such that

\[
1 \oplus \nu \leq \theta + \delta/7 \quad (2.4)
\]

Continuity of \( m: (x, y) \mapsto xy \) at \((0,1)\) produces \( \nu \in \Sigma(0) \) and \( \theta \in \Sigma(1) \) such that

\[
\nu \circ \theta \leq \nu + \delta/7 \quad (2.5)
\]

Since \( r: x \mapsto x^{-1} \) is continuous at \( 1 \), we can find \( \theta \in \Sigma(1) \) such that

\[
\theta^{-1} \leq \theta + \delta/7 \quad (2.6)
\]

Now again applying continuity of \( p \) at \( x = 0 \), we get for \( \theta \in \Sigma(1) \), a \( \nu \in \Sigma(0) \) such that

\[
\nu - \delta/7 \leq -1 \oplus \theta
\]

\[
\Rightarrow 1 \oplus \nu \leq \theta + \delta/7
\]

Then if we can show that

\[
\nu \oplus \nu \leq \mu + \delta/7,
\]

then we are done.

But, first we show the following inequality:

\[
\nu \oplus \nu \leq [(1 \oplus \nu \circ (1 \oplus \nu) \circ (1 \oplus \nu)) \circ (1 \oplus \nu)] \oplus 1
\]

Let \( z \in D \), then

\[
\nu \oplus \nu(z)
\]

\[
= \sup_{x + y = z} \nu \circ (x \wedge \nu \circ y)
\]

\[
= \sup_{x + y = z} \nu \circ (y \wedge \nu \circ x \wedge \nu \circ x)
\]

\[
= \sup_{[1 + y(1 + x)^{-1}]} \nu \circ (y \wedge (1 \oplus \nu \circ x) \circ (1 \oplus \nu \circ x) \circ (1 \oplus \nu \circ x) \circ (1 \oplus \nu \circ x))
\]
This proves the inequality (2.8).

Now we prove that the right side of (2.8) is less than or equal to $\mu + \delta$.

In fact,

$$\nu_3 \circ \nu_2 \leq [(1 \circ \nu_2 \circ (1 \circ \nu_3) \sim) \circ (1 \circ \nu_3) \circ 1$$

$$= [1 \circ \nu_2 \circ (1 \circ \nu_3) \sim \|1 + y(1 \circ x)^{-1}\| (1 \circ \nu_3)(1 + x)$$

$$= [1 \circ \nu_2 \circ (1 \circ \nu_3) \sim ] \circ (1 \circ \nu_3)(z + 1)$$

$$= [1 \circ \nu_2 \circ (1 \circ \nu_3) \sim ] \circ (1 \circ \nu_3) \circ 1(z),$$

i.e.,

$$\nu_3 \circ \nu_2 \leq [1 \circ \nu_2 \circ (1 \circ \nu_3) \sim] \circ (1 \circ \nu_3) \circ 1.$$

In fact, $\nu_3 \circ \nu_2 \leq \mu + \delta$, which proves the continuity of addition. Continuity of addition at $(x, y)$, where $x \neq 0$ follows similarly from the identity $x + y = x(1 + x^{-1}y)$. □

**THEOREM 2.5.** Let $(D, +, \cdot)$ be a commutative division ring.

(i) If $(D, +, \cdot, t(\Sigma_i))_{i \in I}$ is a family of fuzzy neighborhood rings, then $(D, +, \cdot, t(\Sigma) = \sup_{i \in I} t(\Sigma_i))$ is a fuzzy neighborhood ring, where

$$\Sigma(0) = \{\inf_{i \in I_0} \nu_{i_j} : \nu_{i_j} \in \Sigma_j(0); i \in I_0; J_i \in 2^{(J_i)}\} \sim$$

(ii) If $(D, +, \cdot, t(\Sigma_i))_{i \in J}$ is a family of fuzzy neighborhood commutative division rings, then $(D, +, \cdot, t(\Sigma) = \sup_{i \in J} t(\Sigma_i))$ is a fuzzy neighborhood commutative division ring, where

$$\Sigma(0) = \{\inf_{i \in J_0} \nu_{i_j} : \nu_{i_j} \in \Sigma_j(0); i \in I_0; J_i \in 2^{(J_i)}\} \sim$$

**PROOF.** We only prove a part of (ii). Verification of the conditions (i)-(iv) of Theorem 1.6 are straightforward. We check condition (v') (inequality (2.1)).

Let $\mu = \inf_{i \in I_0} \mu_{i_j}$ and $\delta I_0$. Choose $\nu_{i_j}$ satisfying the condition (v') in (2.1) for all $i \in I_0$; and let $\nu = \inf_{i \in I_0} \nu_{i_j}$. Now for any $z \in D$:

$$\nu/(1 \circ \nu)(z) = \nu \circ (1 \circ \nu) \sim (z)$$

$$= \sup_{ab = z} \nu(a) \wedge (1 \circ \nu) \sim (b)$$
DEFINITION 2.6. ([3], [4]). Let \((D, +, \cdot)\) be a commutative division ring and 
\((D, +, \cdot, t(S))\) a fuzzy neighborhood ring. Then a fuzzy set \(\mu \in D\) is called bounded in 
\((D, +, \cdot, t(S))\) if and only if for all \(v \in E(0)\) there exists \(\theta \in E(0)\) such that 
\(\mu \subset \theta \leq v + \delta\).

THEOREM 2.7. In a fuzzy neighborhood commutative division ring \((D, +, \cdot, t(S))\), every 
relatively \(f\)-compact fuzzy set is bounded.

PROOF. Let \(\mu \in D\), \(\delta > 0\) and \(v \in E(0)\). Since multiplication is continuous, for each \(x\), we can 
find a \(\delta \in E(0)\) and \(u \in E(x)\) such that

\[
\sup_{ab = z} \mu(a) \land \sup_{ab = z} \nu(x) = \frac{\sup_{ab = z} \inf_{ab = z} \mu_x(a) \land \sup_{ab = z} \nu_x(x)}{(1 + x)^{-1}} = b
\]

\[
\inf_{ab = z} \sup_{ab = z} \nu_x(a) \land (1 + \nu_x)^{-1}(b) = \inf_{ab = z} \nu_x(1 + \nu_x)^{-1}(z)
\]

\[
\inf_{ab = z} \mu_x(z) + \delta
\]

\[
\nu/1 \ominus \nu \leq \mu + \delta \quad \text{or} \quad \nu \ominus (1 + \nu) \leq \mu + \delta.
\]

\[
\mu \subseteq \theta \leq v + \delta, \quad \text{proving that } \mu \text{ is bounded.} \]
REFERENCES