EFFECT OF GRAVITY ON VISCO-ELASTIC SURFACE WAVES IN SOLIDS INVOLVING TIME RATE OF STRAIN AND STRESS OF HIGHER ORDER

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ABSTRACT. A study is made of the surface waves in a higher order visco-elastic solid involving time rate of change of strain and stress under the influence of gravity. A fairly general equation for the wave velocity is derived. This equation is used to examine various kinds of surface waves including Rayleigh waves, Love waves and Stoneley waves. It is shown that the corresponding classical results follow from this analysis in the absence of gravity and viscosity.

KEY WORDS AND PHRASES: Surface waves, effects of gravity and viscosity.

1. INTRODUCTION

Considerable literature including Bullen [1], Flugge [2] and Stoneley [3] is available on the theory of surface waves in an isotropic homogeneous elastic solid medium. However, the effects of gravity, viscosity and curvature, although important, are not included in the classical problems. Biot [4] has first investigated the effect of gravity on Rayleigh waves on the surface of an elastic solid based on the assumption that gravity produces a type of initial stress of hydrostatic in nature. Subsequently, Biot’s theory has been used by several authors including De and Sengupta [5,6] to study problems of waves and vibrations in solids under the initial stress in various configurations. Further, Sengupta and his associates [7-9] have made an attempt to study the problems of surface waves in solids involving time rate of strain and viscosity. In spite of these studies, relatively less attention has been given to surface wave problems in a higher order visco-elastic solid involving time rate of strain and stress under the influence of gravity. The main purpose of this paper is to study such problems. A fairly general equation for the wave velocity is derived. This equation is utilized to examine various kinds of surface waves including Rayleigh waves, Love waves, and Stoneley waves. It is shown that the corresponding classical results follow from this analysis in the absence of viscosity and gravity.

2. FORMULATION OF THE PROBLEM AND BOUNDARY CONDITIONS

Let $M_1$ and $M_2$ be two homogeneous general visco-elastic solid media involving time rate of strain and stress of higher order in welded contact under the influence of gravity at their common surface of separation. We suppose that the media are separated by a plane horizontal boundary extending to
infinitely great distance from the origin, $M_2$ being above $M_1$. We introduce a set of orthogonal Cartesian coordinate axes $0x_1x_2x_3$ in the semi-infinite isotropic visco-elastic media, with the origin at the common boundary surface and the $x_3$-axis is normal to $M_1$. We consider the possibility of a type of wave travelling in the direction of $0x_1$ in such a manner that the disturbance is largely confined to the neighborhood of the boundary and at any instant all particles on any line parallel to $0x_1$ have equal displacements. Hence the wave is a surface wave and all partial derivatives with respect to the coordinate $x_2$ are zero. Then the components of displacement $u_1$ and $u_3$ at any point may be expressed in the form [1]

$$u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3}, \quad u_3 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi}{\partial x_3};$$

where $\phi$ and $\psi$ are the functions of $x_1, x_3$ and $t$ and

$$\nabla^2 \phi = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \Delta, \quad \nabla^2 \psi = \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}. \tag{2.2ab}$$

Thus the introduction of the functions $\phi$ and $\psi$ enables us to separate out the purely dilational and rotational disturbances associated with the components $u_1$ and $u_3$. The component $u_2$, of course, is associated with purely distortional movement. Thus $\phi, \psi$ and $u_2$ are associated respectively with $P$-waves, $SV$-waves and $SH$-waves, as used by Bullen [1].

The stress-strain relations are

$$D_n \alpha_{ij} = D_\lambda \delta_{ij} + 2D_p e_{ij}, \tag{2.3}$$

where

$$D_n = \sum_{k=0}^{n} \eta_k \frac{\partial^k}{\partial t^k}, \quad D_\lambda = \sum_{k=0}^{\lambda} \lambda_k \frac{\partial^k}{\partial t^k}, \quad D_p = \sum_{k=0}^{\mu} \mu_k \frac{\partial^k}{\partial t^k} \tag{2.4abc}$$

where $\eta_k, \lambda_k$ and $\mu_k$ are the elastic constants and $\eta_k, \lambda_k$ and $\mu_k (k = 1, 2, \ldots, n)$ are the effects of viscosity, $e_{ij}$ is the strain tensor and $\delta_{ij}$ is the Kronecker symbol.

The displacement equations of motion in the higher order general visco-elastic medium, under the influence of gravity, are

$$(D_\lambda + D_\mu) \frac{\partial \Delta}{\partial x_1} + D_\nu \nabla^2 u_1 + \rho g D_n \frac{\partial u_3}{\partial x_1} = \rho D_n \frac{\partial^2 u_1}{\partial t^2}, \tag{2.5}$$

$$D_\mu \nabla^2 u_2 = \rho D_n \frac{\partial^2 u_2}{\partial t^2}, \tag{2.6}$$

$$(D_\lambda + D_\mu) \frac{\partial \Delta}{\partial x_3} + D_\nu \nabla^2 u_3 - \rho g D_n \frac{\partial u_1}{\partial x_3} = \rho D_n \frac{\partial^2 u_3}{\partial t^2}, \tag{2.7}$$

where $\rho, \eta_k, \lambda_k, \mu_k (k = 0, 1, 2, \ldots, n)$ denote the properties of the medium $M_1$ and those with dashes the properties of the medium $M_2$. Substituting (2.1ab) in equations (2.5)-(2.7), we obtain the wave equations in $M_1$ satisfied by $\phi, \psi$ and $u_2$, as

$$\frac{\partial^2 \phi}{\partial t^2} = D_\phi \nabla^2 \phi + g \frac{\partial \psi}{\partial x_1}, \tag{2.8}$$

$$\frac{\partial^2 \psi}{\partial t^2} = D_\psi \nabla^2 \psi - g \frac{\partial \phi}{\partial x_1}, \tag{2.9}$$

$$\frac{\partial^2 u_2}{\partial t^2} = D_\phi \nabla^2 u_2, \tag{2.10}$$

where
and similar relations in $M_2$ with $\rho, \eta, \lambda_1, \mu_1$ replaced by $\rho', \eta', \lambda', \mu'$ and so on (where $k = 0, 1, 2, \ldots, n$).

The boundary conditions are

(i) The components of displacement at the boundary surface between the media $M_1$ and $M_2$ must be continuous at all times and distances.

(ii) The stresses $\sigma_{x1}, \sigma_{x2}, \sigma_{x3}$ are

$$D_1 \sigma_{x1} = D_2 \frac{\partial u_2}{\partial x_1},$$

$$D_1 \sigma_{x2} = D_2 \frac{\partial u_2}{\partial x_2},$$

$$D_1 \sigma_{x3} = D_2 \frac{\partial^2 \tilde{\phi}}{\partial x_3^2} + 2D_3 \left( \frac{\partial \tilde{\phi}}{\partial x_3} \frac{\partial^2 \tilde{\psi}}{\partial x_1 \partial x_3} \right),$$

and similar expressions for $M_2$, across the boundary surface between $M_1$ and $M_2$ must be continuous at all times and distances.

3. SOLUTION OF THE PROBLEM

To solve equations (2.8)-(2.10), we put

$$\phi, \psi, u, \tilde{\phi}, \tilde{\psi}, \tilde{u}_2 = [\tilde{\phi}(x_3), \tilde{\psi}(x_3), \tilde{u}_2(x_3)] e^{i(\omega t - \xi x_3)},$$

for medium $M_1$ and similar solutions for $M_2$, the functions $\tilde{\phi}, \tilde{\psi}, \tilde{u}_2$ being replaced by $\tilde{\phi}', \tilde{\psi}', \tilde{u}_2'$.

Introducing (3.1) in (2.8)-(2.10), we have for the medium $M_1$:

$$\frac{d^2}{dx_3^2} \left( \eta_1^2 V_{11}^2 \right) \tilde{\phi} = -i \omega \eta_1 \tilde{\psi} \eta_1 V_{11}^2,$$

$$\frac{d^2}{dx_3^2} \left( \eta_1^2 V_{12}^2 \right) \tilde{\psi} = -i \omega \eta_1 \tilde{\phi} \eta_1 V_{12}^2,$$

$$\frac{d^2}{dx_3^2} \left( \eta_1^2 V_{13}^2 \right) \tilde{u}_2 = 0,$$

where

$$\eta_1 = \sum_{k=0}^{n} (-i \omega)^k \eta_k, \quad V_{ij}^2 = \sum_{k=0}^{n} (-i \omega)^k V_{i+}^2, \quad V_{ij}^2 = \sum_{k=0}^{n} (-i \omega)^k V_{ij}^2.$$  

Similar relations for $M_2$ can be obtained by replacing $\tilde{\phi}, \tilde{\psi}, \tilde{u}_2, \eta_1, V_{11}^2, \eta_1, V_{12}^2, \eta_1, V_{13}^2, \lambda_2, \mu_2, \rho$ by the same symbols with dashes. Here $\rho, \eta_k, \lambda_k, \mu_k (k = 0, 1, 2, \ldots, n)$ are the physical properties of the medium $M_2$.

We assume that $\phi, \psi$ and $u_3$ represent exponentially decaying solutions in the medium $M_1$ as $x_3 \to \infty$ so that they can be expressed in the form:

$$\phi = A_1 e^{-\gamma \eta_1 \sqrt{-i \omega}} + A_2 e^{-\gamma \eta_1 \sqrt{-i \omega}} e^{i(\omega t - \xi x_3)}$$  

(3.6)
\[ \psi = \left[ B_1 e^{-\sqrt{\eta^2 - \varsigma^2} \xi} + B_2 e^{\sqrt{\eta^2 - \varsigma^2} \xi} \right] e^{i(\nu_1 - \omega t)} \] (3.7)

\[ u_\varphi = \left[ C e^{-\sqrt{\eta^2 - \omega^2 \eta_1 \eta_2} \xi} \right] e^{i(\nu_1 - \omega t)} \] (3.8)

and similar solutions in \( M_j \) can be obtained replacing \( \phi, \psi, u_\varphi, A_1, A_2, B_1, B_2, C, \eta_1, V_{1r}, \varsigma_1, \varsigma_2 \) the same symbols with dashes in solutions (3.6)-(3.8). Here \( \varsigma_j \) and \( \varsigma'_j \) \( (j = 1, 2) \) are respectively the roots of the equations

\[ [\omega^2 - \varsigma^2 V_{1r}^2 \eta_1^*] [\omega^2 - \varsigma^2 V_{1r}^2 \eta_1^*] - \gamma^2 \eta^2 = 0 \] (3.9)

\[ [\omega^2 - \varsigma'^2 V_{1r}^2 \eta_1^*] [\omega^2 - \varsigma'^2 V_{1r}^2 \eta_1^*] - \gamma^2 \eta^2 = 0 \] (3.10)

and

\[ B_1 = \alpha_1 A_1, \quad B_2 = \alpha_2 A_2, \quad B'_1 = \alpha'_1 A_1, \quad B'_2 = \alpha'_2 A_2 \]

where

\[ \alpha_j = i g \eta / (\omega^2 - \varsigma_j^2 V_{1r}^2 \eta_1^*), \quad \alpha'_j = i g \eta / (\omega^2 - \varsigma'_j^2 V_{1r}^2 \eta_1^*) \quad (j = 1, 2) \]

In evaluating quantities like \( \sqrt{\eta^2 - \varsigma^2} \) and \( \sqrt{\eta^2 - \omega^2 \eta_1 \eta_2} \), the root with positive real part must be taken in each case.

Using boundary conditions (i) and (ii), we obtain

\[ [1 - i \alpha_1 Q_1] A_1 + [1 - i \alpha_2 Q_2] A_2 = [1 + i \alpha'_1 Q'_1] A'_1 + [1 + i \alpha'_2 Q'_2] A'_2 \] (3.11a)

\[ C = C' \] (3.11b)

\[ [\alpha_1 + i Q_1] A_1 + [\alpha_2 + i Q_2] A_2 = [\alpha'_1 - i Q'_1] A'_1 + [\alpha'_2 - i Q'_2] A'_2 \] (3.11c)

\[ \rho (V_{1r}^2 \eta_1^*) [2i Q_1 + (1 + Q_1^2) \alpha_1] A_1 + [2i Q_2 + (1 + Q_2^2) \alpha_2] A_2 = \rho (V_{1r}^2 \eta_1^*) [2i Q'_1 + (1 + Q'_1^2) \alpha'_1] A'_1 + [2i Q'_2 + (1 + Q'_2^2) \alpha'_2] A'_2 \] (3.11d)

\[ (\rho \eta_1^*) [V_{1r}^2 (Q_1^2 - 1) + 2V_{1r}^2 (1 - i \alpha_1 Q_1)] A_1 + [V_{1r}^2 (Q_2^2 - 1) + 2V_{1r}^2 (1 - i \alpha_2 Q_2)] A_2 = \rho (\eta_1^*) [V_{1r}^2 (Q_1^2 - 1) + 2V_{1r}^2 (1 + i \alpha'_1 Q'_1)] A'_1 + [V_{1r}^2 (Q_2^2 - 1) + 2V_{1r}^2 (1 + i \alpha'_2 Q'_2)] A'_2. \] (3.11f)

It follows from equations (3.11b) and (3.11e) that \( C = C' = 0 \). Thus there is no propagation of displacement \( u_\varphi \). Hence there are no SH-waves in this case.

From equations (3.11a), (3.11c), (3.11d) and (3.11f), we eliminate the constants \( A_1, A_2, A'_1, A'_2 \), to get equation for the wave velocity in determinant form

\[ |M_{ij}| = 0, \quad (i, j = 1, 2, 3, 4), \] (3.12)

where
and where \( m = 1,2 \).

Equation (3.12) gives the wave velocity for the surface waves in the common boundary, and the strain rate and the stress rate of higher order in the presence of gravity and viscosity are included in (3.12).

4. PARTICULAR CASES

(i) Rayleigh Waves

In order to investigate the possibility of Rayleigh waves, we take the plane boundary as a free surface with \( M_2 \) replaced by a vacuum. Obviously, there are no \( SH \)-waves in this case. In view of (3.11d) and (3.11f), we obtain

\[
\{2iQ_1 + (1 + Q_1^2)\alpha_1\} A_1 + \{2iQ_2 + (1 + Q_2^2)\alpha_2\} A_2 = 0,
\]

and

\[
\{V_{41}^2(Q_1^2 - 1) + 2V_{41}^2(1 - i\alpha_1 Q_1)\} A_1 + \{V_{42}^2(Q_2^2 - 1) + 2V_{42}^2(1 - i\alpha_2 Q_2)\} A_2 = 0.
\]

Eliminating the constants \( A_1 \) and \( A_2 \) from equations (4.1)-(4.2), we get

\[
|M_{ij}| = 0 \quad (i, j = 1, 2)
\]

where

\[
M_1 = [2iQ_1 + (1 + Q_1^2)\alpha_1]; \quad M_2 = [V_{41}^2(Q_1^2 - 1) + 2V_{41}^2(1 - i\alpha_1 Q_1)]; \quad (r = 1, 2).
\]

Equation (4.3) is the required wave velocity equation for visco-elastic Rayleigh waves including the strain rate and stress rate of higher order under gravitational field. When the effects of viscosity and gravity are neglected, this equation reduces to the classical result as discussed by Bullen [1].

(ii) Love Waves

In this case we consider a layered semi-infinite medium in which \( M_2 \) is bounded by two horizontal plane surfaces at a finite distance \( H \)-apart, while \( M_1 \) remains infinite as it was. In this case, we consider the displacement component \( u_z \) only.

For the medium \( M_2 \) we write down the full solution, since the displacement in \( M_2 \) may no longer diminish with the increasing distance from common boundary \( x_3 = 0 \) and for the medium \( M_1 \) the solutions are the same as it was in the general case.

Therefore, for the medium \( M_2 \) we write

\[
u_2 = \left[ C_1 e^{x\sqrt{\eta^2 - \omega^2 \eta_1^2, \eta_1^2}} + C_2 e^{-x\sqrt{\eta^2 - \omega^2 \eta_1^2, \eta_1^2}} \right] e^{i(\eta_1 x - \omega t)},
\]

(4.5)
where the restriction that the real part of $\sqrt{\eta^2 - \omega^2 \eta^2_{\nu} / V_{\nu}^2}$ is positive is not required for $M_2$.

In this case the boundary conditions are

(i) $u_2$ and $\sigma_{32}$ are continuous at $x_3 = 0$ and (ii) $\sigma_{33} = 0$ at $x_3 = -H$.

Applying these boundary conditions and using (3.8) and (4.5), we find

$$C = C'_1 + C'_2$$

(4.6)

$$(-pV_{4s}^2\eta^*_f)\sqrt{\eta^2 - \omega^2 \eta^2_{\nu} / V_{\nu}^2} = [C'_1 - C'_2](pV_{4s}^2\eta^*_i)\sqrt{\eta^2 - \omega^2 \eta^2_{\nu} / V_{\nu}^2}$$

(4.7)

$$C'_1 e^{-i\sqrt{\eta^2 - \omega^2 \eta^2_{\nu} / V_{\nu}^2} x_3} - C'_2 e^{i\sqrt{\eta^2 - \omega^2 \eta^2_{\nu} / V_{\nu}^2} x_3} = 0.$$  

(4.8)

Eliminating $C, C'_1$ and $C'_2$ from the equations (4.6)-(4.8), we obtain

$$(pV_{4s}^2\eta^*_f)\sqrt{1 - c^2 \eta^2_{\nu} / V_{\nu}^2} + (pV_{4s}^2\eta^*_i)\sqrt{(c^2 \eta^2_{\nu} / V_{\nu}^2) - 1} \tan \eta H \sqrt{(c^2 \eta^2_{\nu} / V_{\nu}^2) - 1} = 0$$

(4.9)

where $c = \omega / \eta$. This is the required wave velocity equation for higher order visco-elastic Love waves involving the strain rate and stress rate under the influence of gravity. It is important to note that Love waves are not affected by gravity but by viscosity. When $\eta_0 = 1$ and $\eta^*_f = \eta^*_i = \lambda_i - \lambda = \mu - \mu_i = 0$ ($k = 1, 2, \ldots, n$) then equation (4.9) is in agreement with the corresponding classical result [1] in a perfectly elastic medium.

(iii) Stoneley Waves

In the classical theory, the Stoneley waves are a generalized form of Rayleigh waves propagating along the common boundary of $M_1$ and $M_2$. In the influence of gravity, Stoneley waves along the common boundary of the general visco-elastic solid media $M_1$ and $M_2$ involving the strain rate and stress rate of higher order, are therefore determined by the roots of the frequency equation (3.12). In the absence of these effects, this equation also agrees with the corresponding classical result.

REFERENCES