SOME SUBREGULAR GERMS FOR p-ADIC $Sp_4(F)$

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ABSTRACT. Shalika’s unipotent regular germs were found by the authors in the case of $G = Sp_4(F)$. Next, subregular germs are also desirable, for at least $f(1)$ is constructible in another form for any smooth function $f$ by using Shalika germs. Some of them were not so hard as expected although to find all of them is still not done explicitly.

KEY WORDS AND PHRASES: Shalika germs, unipotent orbits, orbital integrals, subregular germs.


0. INTRODUCTION

Suppose that $G$ is the set of $F$-points of a connected semi-simple algebraic group defined over a $p$-adic field $F$, that $T$ is a Cartan subgroup of $G$, and that $T'$ designate the set of regular elements in $T$. Let $dg$ be a $G$-invariant measure on the quotient space $TG$, and let $C^\infty_r(G)$ be the set of smooth functions. Then it is known that for any $f \in C^\infty_r(G)$ and $t \in T'$ the orbital integral $\int_{T'} f(g^{-1}tg) dg$ is convergent.

Next, let $S_u$ be the set of unipotent conjugacy classes in $G$ and let $dx_0$ be a $G$-invariant measure on $0 \in S_u$. It is also known that $A_0(f) = \int f dx_0$ converges for any $f \in C^\infty_r(G)$.

Shalika, J. A. (see [14], p. 236) says that for any $t \in T'$ sufficiently close to 1, there exist germs $\Gamma_0(t)$ satisfying

$$\int_{T'} f(g^{-1}tg) dg = \sum_{0 \in S_u} \Gamma_0(t) A_0(f).$$

Shalika, J. A., Howe, R., Harish Chandra, Rogawski, J., and others contributed to the establishment of the germ associated to the trivial unipotent class. Recently Repka J. has found regular and subregular germs for $p$-adic $GL_n(F)$ and $SL_n(F)$. The authors also found the regular germs for $p$-adic $Sp_4(F)$ in 1987. In this paper, the authors intend to find some subregular germs associated to some subregular conjugacy classes in $Sp_4(F)$.

Our result may in principle be seen elsewhere, but this paper gives an explicit formula in a special case.

1. NOTATIONS RELATED TO SYMPLECTIC GROUPS

Let $G = Sp_4(F) = \{g \in SL_4(F) : \det g = 1\}$, where $F$ is any $p$-adic field and

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

with $2 \times 2$ identity matrix $I_2$. 
Let \( \sigma \) be the involution on \( SL_4(F) \) defined by \( \sigma(g) = J^{-1}(g) J \) with \( g \in SL_4(F) \). \( G \) may be interpreted as \( SL_4(F)^\prime \). \( G \) may also be expressed as the subgroup of \( SL_4(F) \) generated by all the symplectic transvections whose most general forms are of the type

\[
M(c, \alpha_1, \alpha_2, \beta_1, \beta_2) = \begin{pmatrix}
1 + c\alpha_1\beta_1 & c\alpha_1\beta_2 & -\alpha_1^2c & -\alpha_1\alpha_2c \\
c\alpha_2\beta_1 & 1 + c\alpha_2\beta_2 & -\alpha_2\alpha_3c & -\alpha_2^2c \\
c\beta_1 & c\beta_2 & 1 - c\alpha_1\beta_1 & -c\alpha_1\beta_1 \\
c\beta_1\beta_2 & c\beta_2^2 & -c\alpha_1\beta_2 & 1 - c\alpha_2\beta_2
\end{pmatrix}
\]  

(1.0)

where \( c \neq 0 \) and \( \alpha_i, \beta_i \) are arbitrary variables in a ground field \( F \). So any symplectic element should be of the form

\[
M = \begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}
\]

with \( M_{ij} \in M_4(F) \) satisfying

\[
\begin{align*}
\text{tr}M_{11}M_{22} - \text{tr}M_{21}M_{12} &= \text{tr}M_{22}M_{11} - \text{tr}M_{12}M_{21} = 1 \\
\text{tr}M_{11}M_{22} - \text{tr}M_{21}M_{12} &= \text{tr}M_{22}M_{11} - \text{tr}M_{12}M_{22} = 0
\end{align*}
\]

(1.1)

Hereafter, we let \( F \) be a \( p \)-adic field of odd residual characteristic with ring of integers \( A \); let \( P \) be the maximal ideal of \( A \). Let \( K = Sp(A), K_1 = \{ k \in K : k = id \mod P \} \), and let \( \text{diag}(a, b, a^{-1}, b^{-1}) \) be denoted \( d(a, b) \) for brevity. If \( a = b \), denote \( \text{diag}(a, b, a^{-1}, b^{-1}) \) simply by \( d(a) \). Write \( \text{char}(s) \) for the characteristic polynomial of a matrix \( s \), \( c(s) \) for the pair consisting of the 2nd and 3rd coefficients of the characteristic polynomial of \( s - id \), ignoring the signs that occur in the characteristic polynomial. Conjugating a matrix \( s \) by a matrix \( r \) means to produce \( r^{-1}sr = s' \) unless otherwise stated. Other symbols shall follow the standard convention.

2. UNIPOTENT ORBITS

\( G \) acts on itself by conjugation, so in particular on the set of all unipotent elements.

Referring to [5] §3, we may obtain the following.

**PROPOSITION (2.0).** Any unipotent orbit meets the set of all elements of the form

\[
\begin{pmatrix}
1 & x & \alpha & \beta \\
0 & 1 & \beta - \alpha x & \gamma \\
0 & 0 & 0 & 0 \\
0 & 0 & -x & 1
\end{pmatrix}
\]

(2.1)

where \( \alpha, \beta, \gamma \) and \( x \in F \).

If \( x = 0 \) in (2.1), we may calculate directly to see that the associated unipotent orbits meet the set of non-regular unipotent matrices or the set of regular unipotent matrices which as a \( G \)-set has representatives of the form

\[
\begin{pmatrix}
1 & 1 & 0 & \delta \\
0 & 1 & 0 & \delta \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 1
\end{pmatrix}
\]

with \( \delta \in F^\times(F^\times)^2 \).  

(2.2)

If \( x = 0 \), however, in (2.1), it is not \( GL \)-conjugate to the element with all diagonal and super-diagonal entries equal to 1 and with all other entries equal to zero, i.e., not a regular unipotent element in short. Due to proposition (3.4) in [5] §3, (2.2) represents the orbits of the \( G \)-set consisting of all the regular unipotent elements of \( G \).
On the other hand every subregular unipotent matrix, i.e., the matrix which is $GL(F)$-conjugate to the element with all diagonal entries equal to 1, with superdiagonal entries $(1,0,1)$, and with all other entries equal to zero, must be conjugate to the matrix of the form
\[
\begin{pmatrix}
1 & 0 & \alpha & 0 \\
0 & 1 & 0 & \gamma \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
with $\alpha, \gamma \in F^\times$.

By (1.1), we see easily that for $(a_{ij}) \in G$ and for two subregular unipotent matrices
\[
\begin{pmatrix}
1 & 0 & \alpha & 0 \\
0 & 1 & 0 & \gamma \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \alpha' \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]
as $G$-conjugate if and only if
\[
\begin{aligned}
\frac{\alpha'}{\alpha} a_{11}^2 + \frac{\alpha'}{\alpha} a_{21}^2 &= 1 \\
\frac{1}{\alpha} a_{11} a_{12} + \frac{1}{\alpha} a_{21} a_{22} &= 0 \\
\frac{\gamma'}{\Gamma} a_{12}^2 + \frac{\gamma'}{\Gamma} a_{22}^2 &= 1
\end{aligned}
\]
holds. Without loss of generality, we may put $a_{21} = 0$; substituting $a_{22} = -\frac{\gamma_1 \cdots \gamma_n}{\alpha_{2n}}$ into the last equation in (2.3), we have
\[
\begin{aligned}
\frac{\alpha'}{\alpha} a_{11}^2 + \frac{\alpha'}{\alpha} a_{21}^2 &= 1 \\
\frac{\gamma'}{\alpha} a_{12}^2 + \frac{\gamma'}{\alpha} a_{22}^2 &= 1
\end{aligned}
\]
From this we know that
\[
\begin{pmatrix}
1 & 0 & \alpha & 0 \\
0 & 1 & 0 & \gamma \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
is $G$-conjugate to the following analogous form
\[
\begin{pmatrix}
1 & 0 & \frac{\alpha \gamma}{\alpha x_1^2 + \gamma x_1^2} & 0 \\
0 & 1 & 0 & \frac{\alpha \gamma}{\alpha x_2^2 + \gamma x_2^2} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
where $x_j$'s are arbitrary so that denominators are nonzero. This, however, contains
\[
\begin{pmatrix}
1 & 0 & \alpha x_1^2 & 0 \\
0 & 1 & 0 & \gamma x_2^2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
and
where \( x_i \)'s are nonzero. Hence there exist at most \( 4 + \frac{12}{2} = 10 \) representatives for this form in any case of \( F \). In fact, a trivial computation shows that there are six, seven or eight classes.

Let

\[
\begin{pmatrix}
1 & 0 & \gamma x_i^2 & 0 \\
0 & 1 & \alpha x_i^2 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

with representative pairs \( (x_i, y) \).

Now let \( \tilde{S}(\alpha, \gamma) = (g \in K : g = \tilde{u}(\alpha, \gamma) \mod P) \). We may choose representative pairs \( (\alpha, \gamma) \) with \( |\alpha| = |\gamma| \approx 1 \). By making use of \( \tilde{S}(\alpha, \gamma) \), we intend to compute the Shalika’s germs associated to the unipotent classes of \( \tilde{u}(\alpha, \gamma) \). Any element of \( \tilde{S}(\alpha, \gamma) \) should be of the form

\[
\begin{pmatrix}
1 + p_{11} & p_{12} & \alpha + p_{13} & p_{14} \\
x_{21} & 1 + p_{22} & x_{23} & \gamma + p_{24} \\
x_{31} & x_{32} & 1 + p_{33} & p_{34} \\
p_{41} & x_{42} & x_{43} & 1 + p_{44} \\
\end{pmatrix}
\]

where \( p_{ij} \) are arbitrary in \( P \) and \( x_{ij} \in P \) are rational functions of \( p_{ij} \) with coefficients in \( A \) uniquely determined by (1.1). From this we obviously see that \( \tilde{S}(\alpha, \gamma) = P^{10} \). We shall deal with the relationship between \( \tilde{u}(\alpha, \gamma) \) and \( \tilde{S}(\alpha, \gamma) \) in the upcoming proposition.

Here we shall practice conjugating by a succession of matrices in \( Sp_4(F) \) to simplify \( \tilde{S}(x) \). Let \( s \in \mathbb{S}(x) \). Any matrix of the form (2.4) may be changed into the analogous form with (1,4) entry and (2,3) entry equal by using some matrix of the form (2.1) with \( |\alpha|, |\beta|, |\gamma| \) and \( |x| \approx 1 \). Over any \( p \)-adic field with odd residual characteristic, the Jacobian of these conjugation maps has modulus 1. Conjugating this by the matrix of the form

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
a & 1 & 0 & 0 \\
0 & 0 & 1 & -a \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

with some \( a \in P \) yields the form with (1,4) entry = (2,3) entry = 0. Next conjugating this form by the matrix of the form

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & b & 1 & 0 \\
b & 0 & 0 & 1 \\
\end{pmatrix}
\]

with some \( b \in P \) yields the form

\[
\begin{pmatrix}
* & * & * & 0 \\
* & * & 0 & * \\
* & * & 0 & * \\
* & * & * & * \\
\end{pmatrix}
\]
with *'s as in (2.4), since the Jacobian of each of these conjugation maps has modulus 1. This form is then conjugate to the analogous form with (3,3) entry \(-1\) and with (1,4) entry \(-(2,3) = (3,4)\) entry \(-1\) by the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
c & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

with some \(c \in P\), which may be transformed into the analogous form with (3,3) entry \(-(4,4) = \) entry \(-1\) and with zeros as before by a matrix of the form

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & d & 0 & 1
\end{pmatrix}
\]

with some \(d \in P\). Lastly it may be transformed into the form

\[
\begin{pmatrix}
1 + z_{11} & z_{12} & \tilde{\alpha} & 0 \\
\tilde{\gamma} z_{12} & 1 + z_{22} & 0 & \tilde{\gamma} \\
\frac{1}{\alpha} z_{11} & \frac{1}{\alpha} z_{12} & 1 & 0 \\
\frac{1}{\alpha} z_{12} & \frac{1}{\alpha} z_{22} & 0 & 1
\end{pmatrix}
\]

(2.5)

with some \(z_{ij} \in P\) by conjugating by \(d(e, f)\) with \(e = \sqrt{1 + \frac{p_{15}}{\alpha}}, f = \sqrt{1 + \frac{p_{25}}{\gamma}}\) for some \(p_{15}, p_{25} \in P\). For later use, we let \(\mathcal{S}(\alpha, \gamma)\) be the set of all matrices of the form (2.5), and let \(\bar{P}\) be the composite map of the conjugations which take the form (2.4) to the form (2.5).

3. INTEGRAND FOR SHALIKA'S UNIPOTENT SUBREGULAR GERMS

If any of the form (2.5) may be a unipotent element, either \(z_{12} = 0\) or \(z_{11} = -z_{22}\) is obtained. The former result \(z_{12} = 0\) implies \(z_{11} = 0\), which again yields \(z_{22} = 0\). The latter implies \(\bar{\alpha} z_{11} + \bar{\gamma} z_{12} = 0\), so \(\bar{\gamma} \in (F^*)^2\). Recall that an \(n \times n\) matrix \(u\) is unipotent if and only if \((u - 1)^m = 0\) for some \(m \in Z^+\). So, we get the following considering (2.4) and the proof of [5] Proposition (3.8).

**PROPOSITION (3.0).** Let \(\bar{x} = (\alpha, \gamma)\) be representative pairs with \(-\bar{\gamma} \notin (F^*)^2\). Then the only unipotent orbit intersecting \(\mathcal{S}(\bar{x})\) is the class of \(\bar{u}(\bar{x})\).

Now assume \(\Theta\) to be a nonsquare in \(F^2\) and write \(E^\Theta = F(\sqrt{\Theta})\). Then \(E^\Theta\) being analogous to the unit circle in \(C\), it becomes a compact group under multiplication. More precisely

\[E^\Theta = \{a + b\sqrt{\Theta} : a, b \in F \quad \text{and} \quad a^2 - \Theta b^2 = 1\}.
\]

Supposing that \(T\) be the set of all matrices of the form

\[
\begin{pmatrix}
a & 0 & b & 0 \\
0 & \alpha & 0 & \beta \\
b \Theta_1 & 0 & a & 0 \\
0 & \beta \Theta_2 & 0 & \alpha
\end{pmatrix}
with \(\Theta_1, \Theta_2 \in F(\gamma(F^*))^2\), i.e.
squarefree elements and with \( a^2 - \Theta_1 b^2 - \alpha^2 - \Theta_2 b^2 = 1 \), we may see easily that \( T \) as a subgroup of \( G \) is isomorphic to \( E_1^{*1} \times E_1^{*2} \) both algebraically and topologically, and that \( T \) becomes an elliptic torus as a Cartan subgroup.

According to the Shalika's theorem (see [14] p. 236) as we have mentioned earlier, we have a kind of expansion

\[
\int_{G} f(t^g) d\hat{g} = \sum_{j=1}^{\infty} \Gamma_j(t) \int_{G} f(u^t) d\hat{g}
\]

where \( \{u_i\} \) is a finite set of representatives of the unipotent orbits, \( f \in C_c(G) \), and \( t \) is any regular element sufficiently close to the identity, although "how close" depends on \( f \). Here the functions \( \Gamma_j \) called Shalika's germs do not depend on \( f \), but depend on a maximal torus \( T \).

We intend to compute the functions \( \Gamma_{aG}(t) \) corresponding to the element \( \bar{u}(x) \) of \( \mathfrak{g} \) by letting \( f = \chi_{aG} \) be the characteristic function of the set \( \mathfrak{g}(x) \) defined in \( \mathfrak{g} \). Thanks to proposition (3.0), the integrals on the right hand side of (3.1) all vanish in the case of \( f = \chi_{aG} \) with \( \bar{a} = (a, \bar{a}) \) and \( -\frac{2}{i} \notin (F^*)^2 \) except for that corresponding to \( \bar{u}(x) \). This facilitates for us to compute the germs sought, but it may not be easy to calculate the others, i.e., those for the pairs with \( -\frac{2}{i} \in (F^*)^2 \). This note deals with the former cases only.

4. CHANGE OF VARIABLES AND JACOBIANS

Let \( s \) be a regular element of \( T \) sufficiently close to the identity; write \( t = x + d \), and assume that \( t \) is an element such that the nontrivial coefficients of the characteristic polynomial of \( t - id \) are in \( P \), i.e., \( c(t) \in P^2 \) according to our convention. By the way \( char(t^t) = char(t) = det(t - \lambda \cdot 1) = \lambda^4 - 2(a + \alpha) \lambda^3 + 2(1 + 2a \alpha) \lambda^2 - 2(a + \alpha) \lambda + 1 \), where \( a \) and \( \alpha \) refer to the entries in the matrix in §3.

On the other hand, the characteristic polynomial of a matrix \( s \) in the form (2.6) turns out to be

\[
char(s) = \lambda^4 + \lambda^3(-4 - z_{11} - z_{22}) + \lambda^2\left(6 + 2z_{11} + 2z_{22} + z_{11}z_{22} - z_{12}^2 + \frac{\bar{\gamma}}{\alpha}\right) + \lambda(-4 - z_{11} - z_{22}) + 1.
\]

So we obviously see that \( c(s) \in P^2 \). In case that \( s \) and \( t \) are conjugate, the corresponding coefficients of \( char(s) \) and \( char(t) \) must be the same, thus the following must hold:

\[
\begin{align*}
z_{22} &= 2(a + \alpha) - 4 - z_{11} \\
z_{11}z_{22} - \frac{\bar{\gamma}}{\alpha}z_{12}^2 + 6 - 2(1 + 2a \alpha) + 4(a + \alpha) - 8 &= 0
\end{align*}
\]

\[
\begin{align*}
z_{22} &= 2(a + \alpha) - 4 - z_{11} \\
z_{11}z_{22} - \frac{\bar{\gamma}}{\alpha}z_{12}^2 - 4(a - 1)(\alpha - 1) &= 0
\end{align*}
\]

\[
\begin{align*}
z_{22} &= 2(a + \alpha) - 4 - z_{11} \\
z_{11}^2 - 2((a - 1) + (\alpha - 1))z_{11} + 4(a - 1)(\alpha - 1) + \frac{\bar{\gamma}}{\alpha}z_{12}^2 &= 0.
\end{align*}
\]

The last equations are solvable if and only if \( (a - \alpha)^2 - \frac{1}{\alpha}z_{12}^2 \in (F^*)^2 \cap (0) \).

For any given matrix \( s \) of the form (2.5) subject to (4.0), we are going to determine whether we may find \( g \in G \) satisfying \( t^g = s \). But we see easily by trivial computation that: in case that
s ∈ \overline{S}_{d}(\alpha, \gamma) is any element with the property \( z_{12} \neq 0 \), and with \((a - \alpha)^2 - \frac{7}{9} z^2_{12}\) square, there exists \( g \in G \) s.t.

\[
t^e = s \in \overline{S}(x) \quad \text{for} \quad x = (\alpha, \gamma)
\]

\[
\Leftrightarrow \frac{2b\alpha}{p(\alpha - a) \in N^{E^0}_{\mathfrak{r}}(\{E\})^e}
\]

\[
\text{and} \quad \frac{2b\alpha}{Q(\alpha - a) \in N^{E^0}_{\mathfrak{r}}(\{E\})^e}
\]

(4.1)

For a fixed regular \( t \in T \), let \( c' : TG \to G \) be the continuous map given by \( c'(g) = t^e \). Put \( \overline{G}(t) = (c')^{-1}(\overline{S}(x)) \). The orbital integral of \( f = \chi_{\overline{G}} \) over the conjugacy class of \( t \) is just the measure of \( \overline{G}(t) \). Define a mapping \( P' : \overline{S} \times P^1 \to P \times P^1 \) via \( P'((t_{11}, z_{12}, z_{22}), \ldots) = ((t_{12}) \ldots) \), which is obviously a projection. Now we construct the following composite map:

\[
(T \times G) \ni \overline{G}(t) \to \overline{S}(x) \to \overline{S} \times P^1 \to P \times P^1.
\]

Fig. 1

Here the middle arrow \( \hat{P} \) in Fig. 1 arises as a homeomorphism which has shown up in \( \S \). Due to the above description, if (2.5) satisfies (4.1), this composite map is bijective except at \( z_{12} = 0 \) and at \( z_{12} \) which does not make \((a - \alpha)^2 - \frac{7}{9} z^2_{12}\) square. We want to find out the composite map's Jacobian so that we may compute the measure of \( \overline{G}(t) \).

Let \( U \) be a neighborhood of a fixed \( t \in T' \cap K_{1} \) chosen so that no two elements of \( U \) are conjugate. Let \( A \subset T' \times TG \) be an open set \( A = \{(t, g) : t \in U, t^e \in \overline{S}(x)\} \). Construct the following commuting diagram.

The upper left mapping \( c^T \) is just the conjugation map taking \((t, g) \in T' \times TG\) to \( t^e = g^{-1}tg \) and \( B = c \times \text{id}(A) \). The middle vertical map \( c \times P' \circ \hat{P} \) denotes \( c \times P' \circ \hat{P}(s) = (c(s), P' \circ \hat{P}(s)) \forall s \in \overline{S}(x) \). Specifically \( c(s) = (c_1, c_2) \), where \( c_1 = \text{trace}(s - 1) \) and \( c_2 = \text{the coefficients of } \lambda^2 \text{ appearing in } \left| s - 1 - \lambda \cdot 1 \right| \). For \((s_3, p_1, \ldots, p_t) \in \overline{S} \times P^t\), the opposite diagonal map \( P'' \) is defined as \( P''((s_3, p_1, \ldots, p_t) = (c(s_3), z_{12}, p_{1}, \ldots, p_t) \).

Next we shall discuss the Jacobians of these maps. The Jacobian of the map \( c^T \) is just \( D(t) = \text{det}(\text{id} - A(t))_{\mathfrak{g}_G} \) where \( \mathfrak{g} \) and \( t \) are the associated Lie algebras of \( G \) and \( T \), respectively (see [14] p. 231). It is not hard to know \( |J(c)| = |(a - \alpha)\sqrt{a^2 - 1} \sqrt{a^2 - 1}| \). Moreover, since \( |J(\hat{P})| = |J(P'')| = 1 \), we have

\[
|J(P' \circ \hat{P} \circ c')| = \left| \frac{D(t)}{\sqrt{a^2 - 1} \sqrt{a^2 - 1}} \right|
\]

(4.2)
As in [5] §5, we have \(|D(t)| = \sqrt{a^2 - 1} \sqrt{\alpha^2 - 1}(a - \alpha)|^2\) and \(|J(c \times P' \circ \hat{P})| = 1\). Hence
\[
|J(P' \circ \hat{P} \circ c')| = |D(t)||a - \alpha|\sqrt{a^2 - 1}\sqrt{\alpha^2 - 1}| = |D(t)|^{1/2}
\]

5. ORBITAL INTEGRALS WITH NORMALIZATION OF MEASURES

We take the natural additive measure \(dx\) on \(F\) so that \(A\) has measure 1. As \(T = E_1^{\theta_1} \times E_1^{\theta_1}\) and \((E_1^{\theta_1})^* / F^* \supset E_1^{\theta_1}/\{\pm 1\}\), choices of measures on \((E_1^{\theta_1})^*\) and \(F^*\) determine a choice of measure on \(E_1^{\theta_1}\).

On \((E_1^{\theta_1})^*\) we may take the corresponding measure \(d'x = \frac{dx}{|x|^{1/2}}\), and on \(F^* \cup d's = \frac{dx}{|x|}\). Now select the measure on \(G\) whose restriction to \(K\) is an extension of the standard measure of \(\widetilde{S}(\overline{x}) = P^{10}\). Since \(|J(c \circ P' \circ \hat{P})| = 1\), Haar measure of \(\widetilde{S}(\overline{x})\) must be the same as that of \(P^{10}\). A choice of measure on \(T \cap G\) depends on that of \(G\) and \(T\) which also gives the natural measure on \(K\) and \(\widetilde{S}(\overline{x})\).

Now recall
\[
\overline{P} = 2 - 2\alpha + z_{22}, \quad \overline{Q} = 2 - 2\alpha + z_{22},
\]
i.e., explicitly
\[
\overline{P} = \alpha - \alpha \pm \sqrt{(a - \alpha)^2 - \frac{\overline{\gamma}}{\alpha} z_{12}^2}
\]
\[
\overline{Q} = \alpha - \alpha \pm \sqrt{(a - \alpha)^2 - \frac{\overline{\gamma}}{\alpha} z_{12}^2}.
\]

We put
\[
X(a, \alpha, \overline{\alpha}, \overline{\gamma}) = \left\{ z_{12} \in P : a - \alpha + \sqrt{(a - \alpha)^2 - \frac{\overline{\gamma}}{\alpha} z_{12}^2} \in N_F^{\theta_1}(E_1^{\theta_1}) \right\},
\]
\[
X'(a, \alpha, \overline{\alpha}, \overline{\gamma}) = \left\{ z_{12} \in P : a - \alpha - \sqrt{(a - \alpha)^2 - \frac{\overline{\gamma}}{\alpha} z_{12}^2} \in N_F^{\theta_1}(E_1^{\theta_1}) \right\},
\]
\[
Y(a, \alpha, \overline{\alpha}, \overline{\gamma}) = \left\{ z_{12} \in P : a - \alpha + \sqrt{(a - \alpha)^2 - \frac{\overline{\gamma}}{\alpha} z_{12}^2} \in N_F^{\theta_1}(E_1^{\theta_1}) \right\},
\]
\[
Y'(a, \alpha, \overline{\alpha}, \overline{\gamma}) = \left\{ z_{12} \in P : a - \alpha - \sqrt{(a - \alpha)^2 - \frac{\overline{\gamma}}{\alpha} z_{12}^2} \in N_F^{\theta_1}(E_1^{\theta_1}) \right\},
\]
\[
\overline{X}(a, \alpha, \overline{\alpha}, \overline{\gamma}) = \left\{ z_{12} \in P : a - \alpha + \sqrt{(a - \alpha)^2 - \frac{\overline{\gamma}}{\alpha} z_{12}^2} \in F^n N_F^{\theta_1}(E_1^{\theta_1}) \right\},
\]
\[
\overline{X}'(a, \alpha, \overline{\alpha}, \overline{\gamma}) = \left\{ z_{12} \in P : a - \alpha - \sqrt{(a - \alpha)^2 - \frac{\overline{\gamma}}{\alpha} z_{12}^2} \in F^n N_F^{\theta_1}(E_1^{\theta_1}) \right\},
\]
\[
\overline{Y}(a, \alpha, \overline{\alpha}, \overline{\gamma}) = \left\{ z_{12} \in P : a - \alpha + \sqrt{(a - \alpha)^2 - \frac{\overline{\gamma}}{\alpha} z_{12}^2} \in F^n N_F^{\theta_1}(E_1^{\theta_1}) \right\},
\]
\[
\overline{Y}'(a, \alpha, \overline{\alpha}, \overline{\gamma}) = \left\{ z_{12} \in P : a - \alpha - \sqrt{(a - \alpha)^2 - \frac{\overline{\gamma}}{\alpha} z_{12}^2} \in F^n N_F^{\theta_1}(E_1^{\theta_1}) \right\}.
\]

Let \(\overline{m}\) be the Haar measure function on these sets. We have then the following orbital integrals.
PROPOSITION 5.0. (i) If \( \frac{2a_{11}}{a_{11}} \in \mathbb{N} \) and \( \frac{2a_{22}}{a_{22}} \notin \mathbb{N} \), then

\[
\int_{\mathcal{O}(\mathbb{Q})} \chi_{\mathcal{O}(\mathbb{Q})}(t^x) dg = \overline{m}((X \cap Y) \cup (X' \cap Y')) \times q^{-x} \times |D(t)|^{-1/2}.
\]

(ii) If \( \frac{2a_{11}}{a_{11}} \in \mathbb{N} \) and \( \frac{2a_{22}}{a_{22}} \notin \mathbb{N} \), then

\[
\int_{\mathcal{O}(\mathbb{Q})} \chi_{\mathcal{O}(\mathbb{Q})}(t^x) dg = \overline{m}((X \cap Y) \cup (X' \cap Y')) \times q^{-x} \times |D(t)|^{-1/2}.
\]

(iii) If \( \frac{2a_{11}}{a_{11}} \notin \mathbb{N} \) and \( \frac{2a_{22}}{a_{22}} \notin \mathbb{N} \), then

\[
\int_{\mathcal{O}(\mathbb{Q})} \chi_{\mathcal{O}(\mathbb{Q})}(t^x) dg = \overline{m}((X \cap Y) \cup (X' \cap Y')) \times q^{-x} \times |D(t)|^{-1/2}.
\]

(iv) If \( \frac{2a_{11}}{a_{11}} \notin \mathbb{N} \) and \( \frac{2a_{22}}{a_{22}} \notin \mathbb{N} \), then

\[
\int_{\mathcal{O}(\mathbb{Q})} \chi_{\mathcal{O}(\mathbb{Q})}(t^x) dg = \overline{m}((X \cap Y) \cup (X' \cap Y')) \times q^{-x} \times |D(t)|^{-1/2}.
\]

PROOF. We have already seen the Jacobian of \( P \circ \hat{P} \circ c' \) is just \( |D(t)|^{1/2} \) and that the measure of \( P \) is fixed to be \( q^{-1} \). So we have our result considering the above remark and (4.1).

Now we must look for the orbital integral over the conjugacy class of \( \overline{u}(\alpha, \gamma) \). To see this we need to specify the measure on the centralizer \( Z(\overline{u}(1,1)) \). Any element of \( Z(\overline{u}(1,1)) \) should be of the form:

\[
\begin{pmatrix}
    a_{11} & \pm \sqrt{1-a_{11}}^2 & a_{13} & a_{14} \\
    \pm \sqrt{1-a_{11}}^2 & a_{11} & a_{23} & a_{13} - \frac{a_{22}a_{11} \pm a_{11}a_{14}}{\sqrt{1-a_{11}^2}} \\
    0 & 0 & a_{11} & \pm \sqrt{1-a_{11}^2} \\
    0 & 0 & \mp \sqrt{1-a_{11}^2} & a_{11}
\end{pmatrix}
\]

if \( a_{11} \neq \pm 1 \) and \( \sqrt{1-a_{11}^2} \in F \),

or

\[
\begin{pmatrix}
    1 & 0 & a_{13} & \mp a_{23} \\
    0 & \mp 1 & a_{23} & a_{24} \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & \mp 1
\end{pmatrix}
\]

if \( a_{11} = \pm 1 \).

By the way \( \overline{u}(\alpha, \gamma) = d(\sqrt{\alpha}, \sqrt{\gamma})\overline{u}(1,1)d(\sqrt{\alpha^{-1}}, \sqrt{\gamma^{-1}}) \) implies that

\[
Z(\overline{u}(\alpha, \gamma)) = Z(d(\sqrt{\alpha}, \sqrt{\gamma}) \cdot \overline{u}(1,1) \cdot d(\sqrt{\alpha^{-1}}, \sqrt{\gamma^{-1}}))
\]

\[
= d(\sqrt{\alpha}, \sqrt{\gamma}) \cdot Z(\overline{u}(1,1)) \cdot d(\sqrt{\alpha^{-1}}, \sqrt{\gamma^{-1}})
\]

We decompose \( G \) into the form

\[
G = B_{(\alpha, \gamma)} \cdot K = Z(\overline{u}(\alpha, \gamma)) \cdot \hat{P} \cdot K, \quad \text{where} \quad B_{(\alpha, \gamma)} = Z(\overline{u}(\alpha, \gamma)) \cdot \hat{P}
\]

and \( \hat{P} = \{ d(b_{11}) : \forall b_{11} \in F^* \} \).
Hence the integral over \( \mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \) may be replaced by an integral over \( \{\mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \circ \mathcal{P} \} \) \( \times K \), and this coset space may be represented by a subset of \( P \cdot K \), the measure of \( P \) being just \( d^*h_1 \), and \( d\gamma \) being an appropriate Haar measure of \( \mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \). Next, consider the following integral. For any \( f \in C^\infty_c(\mathcal{G}), \)

\[
\int_J f(g) dg = \int_{\mathcal{Z}(\mathcal{G}, \mathcal{H})} f(zg) dz dg = \int_K \int_{\mathcal{Z}(\mathcal{G}, \mathcal{H})} f(zg) \cdot dz \cdot \frac{dp \cdot dk}{\Delta_{\mathcal{G}, \mathcal{H}}(p)}
\]

where \( \Delta \) arises because \( \mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \times \mathcal{P} \times K \rightarrow \mathcal{G} \) given by \( (z, p, k) \rightarrow z \cdot p \cdot k \) is not a topological isomorphism. We may figure out the constant \( \Delta \) by calculating the measure of \( K \). The modular function being trivial on \( \mathcal{P} \cap K \),

\[
\int_K f(g) dg = \int_{\mathcal{Z}(\mathcal{G}, \mathcal{H}) \cap K} f(zg) dz dg = \int_K \int_{\mathcal{Z}(\mathcal{G}, \mathcal{H}) \cap K} f(zg) \cdot dz dp dk
\]

The inner integrals must be the same after setting \( f = \chi_K \), the characteristic function of \( K \); so deleting these, we obtain

\[
\int_{\mathcal{Z}(\mathcal{G}, \mathcal{H})} 1 \cdot d\gamma = \int_K \int_{\mathcal{P} \cap K} \Delta \cdot dp dk = \int_{\mathcal{Z}(\mathcal{G}, \mathcal{H})} \int_K \int_{\mathcal{P} \cap K} \Delta \cdot dp \cdot dz \cdot dg
\]

So, we have \( \Delta = \left(1 - \frac{1}{q} \right)^{-1} \cdot m(\mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \cap K) \). Hence the quotient measure of \( \mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \mathcal{G} \) is obtained by writing \( g = pk \) with \( p \in \mathcal{P} \), \( k \in K \) and putting \( dg = \left(1 - \frac{1}{q} \right)^{-1} \cdot m(\mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \cap K) \cdot d\gamma \). Since \( B(\alpha, \gamma) \) is not unimodular although \( G, K, \mathcal{P} \) and \( \mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \) are unimodular.

**PROPOSITION 5.1.** With the assumption of measures normalized as above, we have

\[
\int_{\mathcal{Z}(\mathcal{G}, \mathcal{H})} \chi_{\mathcal{S}(\mathcal{G}, \mathcal{H})} \mathcal{Z}(\widetilde{u}(\alpha, \gamma)) d\gamma = q^{-7}
\]

**PROOF.** The decomposition \( G = B(\alpha, \gamma) \cdot K \) assures that any element conjugate to \( \widetilde{u}(\alpha, \gamma) \) is determined by \( g = pk \) with \( p \in \mathcal{P} \) and \( k \in K \). So, we have

\[
\int_{\mathcal{Z}(\mathcal{G}, \mathcal{H})} \chi_{\mathcal{S}(\mathcal{G}, \mathcal{H})} \mathcal{Z}(\widetilde{u}(\alpha, \gamma)) d\gamma = \int_K \int_{\mathcal{P} \cap K} \chi_{\mathcal{S}(\mathcal{G}, \mathcal{H})} \mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \cdot d\gamma.
\]

By the way \( p^{-1} \mathcal{Z}(\widetilde{u}(\alpha, \gamma)) p - ksk^{-1} \) for \( k \in K, s \in \mathcal{S}(\alpha, \gamma) \) implies that \( p \in \mathcal{P} \cap K \). Hence it is not difficult to see that \( k = p^{-1} k' \) with \( k' \in \mathcal{Z}(\mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \cap K) \) if and only if \( \mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \in \mathcal{S}(\alpha, \gamma) \). Since the modular function is 1 for \( p \in K \), we obtain

\[
\int_{\mathcal{Z}(\mathcal{G}, \mathcal{H})} \chi_{\mathcal{S}(\mathcal{G}, \mathcal{H})} \mathcal{Z}(\widetilde{u}(\alpha, \gamma)) d\gamma = \int_{\mathcal{Z}(\mathcal{G}, \mathcal{H}) \cap K} \int_{\mathcal{P} \cap K} \left(1 - \frac{1}{q} \right)^{-1} \cdot m(\mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \cap K) \cdot d\gamma.
\]

Since the measure of \( \mathcal{P} \cap K \) is \( 1 - \frac{1}{q} \) and the measure of \( \mathcal{Z}(\mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \cap K) \cdot K_1 \) is just \( q^{-7} \cdot m(\mathcal{Z}(\widetilde{u}(\alpha, \gamma)) \cap K) \), we have

\[
\int_{\mathcal{Z}(\mathcal{G}, \mathcal{H})} \chi_{\mathcal{S}(\mathcal{G}, \mathcal{H})} \mathcal{Z}(\widetilde{u}(\alpha, \gamma)) d\gamma = q^{-7}
\]

as required.
Finally, we combine everything, in particular propositions (5.0) and (5.1) to yield our main result. Notice that $\Theta_i$ may belong to three nontrivial residue classes mod $(F^*)^2$.

**THEOREM 5.2.** Suppose that we are given an elliptic torus $T$ as in §3. Then the Shalika’s unipotent subregular germs for $G$ in the case of $-\frac{a}{y} \notin (F^*)^2$ are obtained case by case as follows:

(i) If $\frac{2a}{a-a} \in N_F^{e_i}((E_i))$ and $\frac{2a}{a-a} \notin N_F^{e_i}((E_i))$, then

$$\Gamma_{(\mathcal{A})} = \overline{m}((X \cap Y) \cup (X' \cap Y')) \times |D(t)|^{-\frac{1}{2}}.$$  

(ii) If $\frac{2a}{a-a} \in N_F^{e_i}((E_i))$ and $\frac{2a}{a-a} \notin N_F^{e_i}((E_i))$, then

$$\Gamma_{(\mathcal{A})} = \overline{m}((X \cap Y) \cup (X' \cap Y')) \times |D(t)|^{-\frac{1}{2}}.$$  

(iii) If $\frac{2a}{a-a} \notin N_F^{e_i}((E_i))$ and $\frac{2a}{a-a} \in N_F^{e_i}((E_i))$, then

$$\Gamma_{(\mathcal{A})} = \overline{m}((X \cap Y) \cup (X' \cap Y')) \times |D(t)|^{-\frac{1}{2}}.$$  

(iv) If $\frac{2a}{a-a} \notin N_F^{e_i}((E_i))$ and $\frac{2a}{a-a} \notin N_F^{e_i}((E_i))$, then

$$\Gamma_{(\mathcal{A})} = \overline{m}((X \cap Y) \cup (X' \cap Y')) \times |D(t)|^{-\frac{1}{2}}.$$ 

**REFERENCES**