FIXED POINT THEOREMS FOR NON-SELF MAPS

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(Received July 29, 1992)

ABSTRACT. Suppose \( f: C \to X \) where \( C \) is a closed subset of \( X \). Necessary and sufficient conditions are given for \( f \) to have a fixed point. All results hold when \( X \) is complete metric space. Several results hold in a much more general setting.

KEY WORDS AND PHRASES. Commuting, compatible, \( d \)-complete topological spaces, fixed points, non-self maps, pairs of mappings.

1991 AMS SUBJECT CLASSIFICATION CODE. 47H10, 54H25.

1. INTRODUCTION.

Fixed point theorems for non-self maps are unusual. We surely require that \( C \cap f(C) \) is non-empty. \( f(x) = x + 1 \) for \( X \) in \([0,1]\) is a linear isometry from the compact space \([0,1]\) into the compact space \([0,2]\) but \( f \) is fixed point free. The mapping \( f(x) = x + \frac{1}{2} \) for \( x \) in \([1,\infty)\) is a continuous mapping from \([1,\infty)\) into \([0,\infty)\). It is fixed point free but \( |f(x) - f(y)| < |x - y| \) for \( x \neq y \).

THEOREM (Brouwer [1]). If \( E \) is a non-empty convex compact subset of \( E^n \) and \( f: E \to E \) is continuous, then \( f(x) = x \) for some \( x \) in \( E \).

2. RESULTS.

THEOREM 1. Let \( C \) be a closed subset of a complete metric space \( X \) and suppose \( f \) maps \( C \) onto \( X \) or \( f \) maps \( C \) into \( X \) with \( C \subset f(C) \). If for some \( k > 1 \), \( d(f(x), f(y)) \geq k \ d(x,y) \) for every \( x, y \) in \( C \), then \( f \) has a unique fixed point in \( C \).

PROOF. Clearly, \( f \) is one-to-one. Let \( g = f^{-1} \) restricted to \( C \). Now \( g \) maps \( C \) into \( C \). For \( x, y \) in \( C \), \( d(x,y) = d(f(gx), f(gy)) \geq k \ d(g(x), g(y)) \) or \( d(g(x), g(y)) \leq \frac{1}{k} \ d(x,y) \) and \( 0 < \frac{1}{k} < 1 \). \( g \) has a unique fixed point fromBanach’s fixed point theorem. But \( f(x_0) = f(g(x_0)) = x_0 \). If \( x_1 = f(x_1) \), then \( g(x_1) = g(f(x_1)) = x_1 \) and \( c_1 = x_0 \).

The above result suggests that one should consider non-self maps that satisfy \( C \subset f(C) \). It is well known that a continuous function from an arc onto a containing arc must have a fixed point. \([0,1]\) or any homeomorphic image is called an arc. Thus Brouwer’s theorem extends to the case \( C \subset f(C) \) for \( n = 1 \). In [7], Sam Nadler showed that for \( n \geq 2 \) Brouwer’s theorem does not extend. For \( n \geq 2 \), let \( A \) and \( B \) be closed balls in \( E^n \) with \( A \subset B \) and \( A \neq B \). He showed that there exists \( f \) and \( g \) such that:

(a) \( f: A \to B \) where \( f \) is continuous, onto, \( f(\partial A) = B \), and \( f \) is fixed point free,

(b) \( g: A \to B \) where \( g \) is continuous, onto, \( g^{-1}(\partial B) = \partial A \), and \( g \) is fixed point free.
THEOREM 2. Let $C$ be a closed bounded, and convex subset of a uniformly convex Banach space and suppose $f$ maps $C$ onto $X$ or $f$ maps $C$ into $X$ with $C \subset f(C)$. If for every $x, y$ in $C$

\[ \| f(x) - f(y) \| \geq \| x - y \|, \]

then $f$ has a fixed point in $C$.

PROOF. Clearly, $f$ is one-to-one. Let $g = f^{-1}$ restricted to $C$ and observe that

\[ \| g(x) - g(y) \| \leq \| x - y \| \]

where $g: C \to C$. From Kirk's theorem [6], $g$ has a fixed point $x_0$ in $C$. Clearly, $f(x_0) = x_0$.

The following is an example of a mapping $f$ that takes a closed, bounded, and convex subset $C$ of a Banach space $X$ into $X$ where $C \subset f(C)$, $\| f(x) - f(y) \| = \| x - y \|$ for all $x, y \in C$, and $f$ has no fixed points.

EXAMPLE 1. Let $X$ be the space of sequences which converge to zero with

\[ \| x \| = \sup_{n} |x_n| \]

for $x$ in $X$. Let $C = \{ x \in X: \| x \| = 1 \text{ and } x_0 = 1 \}$. $C$ is closed, bounded, and convex. Define $f: C \to X$ by $f(x) = y$ where $y_n = x_{n+1}, n = 0, 1, 2, \cdots$. $f(x) - f(y) = x - y$ and $f$ is linear. To see that $C \subset f(C)$ consider the following. For $z \in C$, define $r$ to be the sequence where $r_0 = 1$ and $r_n = x_{n-1}$, $n = 1, 2, 3, \cdots$. Then $r \in C$, and $f(r) = z$ so $C \subset f(C)$. If $s = \{1, 0, 0, \cdots, \}, s \in C$ but $f(s) = \{0, 0, 0, \cdots\} \notin C$. Hence $C \neq f(C)$. If $f(x) = x$ for some $x$ in $C$,

then $x_n = x_{n+1}$ for $n = 0, 1, 2, \cdots$. Since $x_0 = 1, x_n = 1$ for all $n$ and $x \notin C$. Therefore, $f$ does not have a fixed point in $C$.

The following example shows that Banach's fixed point theorem does not generalize to non-self maps.

EXAMPLE 2. Let $X = (R, R)$ with $\| f \| = \sup_{t \in R} |f(t)|$ for $f \in X$. Let $C = \{ f \in X: f(t) = 0 \text{ for all } t \leq 0 \text{ and } \lim_{t \to \infty} f(t) \geq 1 \}$. $C$ is a closed and convex subset of $X$. Define $T: C \to X$ by $(Tf)(t) = \frac{1}{2} f(t + 1)$. To see that $C \subset T(C)$ consider the following. For $f$ in $C$ set $g(t) = 2f(t - 1)$ and $g(t) = 0$ for $t \geq 0$ since $t - 1 < 0$ and $f(t) = 0$ for all $t \leq 0$.

\[ \lim_{t \to \infty} g(t) = \lim_{t \to \infty} 2 f(t - 1) \geq 2. \]

Thus $g \in C$ and $(Tg)(t) = f(t)$. Hence $C \subset T(C)$. Let $f(t)$ be defined as $0$ if $t \leq 0$, $t$ if $0 < t < 1$, and $1$ if $t \geq 1$. Then $f \in C$. Now $(Tf)(t)$ is $0$ if $t \leq -1$, $\frac{1}{2} (t + 1)$ if $-1 < t < 0$, and $\frac{1}{2}$ if $t \geq 0$. Therefore, $Tf \notin C$ and $C \neq T(C)$. For $f, g \in C$, $\| Tf - Tg \| = \frac{1}{2} \| f - g \|$. If $Tf = f$ for some $f \in C$, then $f(t) = \frac{1}{2} f(t + 1)$ and it follows that $f(n) = 0$ for all integers $n$. Hence $\lim_{t \to \infty} f(t) \notin C$ and $C \neq T(C)$. Therefore $T$ does not have a fixed point in $C$. Note that $T$ is linear, one-to-one, and $T(C)$ is closed.

We now turn to finding necessary and sufficient conditions for a non-self map to have a fixed point. Then it becomes clear that $C \subset f(C)$ is a natural assumption.

Let $(X, t)$ be a topological space and $d: X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. $X$ is said to be $d$-complete if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that the sequence $\{x_n\}$ is convergent in $(X, t)$. These spaces include complete (quasi) metric spaces and $d$-complete (symmetric) semi-metric spaces. In [2] and [3] several basic metric space fixed point theorems were extended to this setting. $f: X \to X$ is $w$-continuous at $x$ if $x_n \to x$ as $n \to \infty$ implies $f(x_n) \to f(x)$ as $n \to \infty$.

The following definition was given by G. Jungck in [5].

DEFINITION 1. Two maps $f$ and $g$ are compatible if, for any sequence $\{x_n\}$ such that $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = t$ it follows that $\lim_{n \to \infty} d(f(gx_n), g(fx_n)) = 0$. Commuting maps are compatible but the converse is false.

DEFINITION 2. Given a map $f$, a map $g$ is compatible with $f$, if for any sequence $\{x_n\}$ such that $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = t$ it follows that $\lim_{n \to \infty} f(g(x_n)) = g(t)$.

REMARK 1. If $f$ and $g$ are $w$-continuous and $(X, d)$ is a metric space, then, using definition
2, \( f \) is compatible with \( g \) is equivalent to \( g \) is compatible with \( f \). In this case, we say that \( f \) and \( g \) are compatible.

**Proof.** Assume \( f \) and \( g \) are \( w \)-continuous and that \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = t \) implies \( \lim_{n \to \infty} f(g(x_n)) = g(t) \). If we are in a metric space,

\[
d(f(gx_n), f(fx_n)) \leq d(f(gx_n), g(t)) + d(g(t), g(fx_n))
\]

and

\[
d(g(fx_n), f(t)) \leq d(g(fx_n), f(gx_n)) + d(f(gx_n), f(t)).
\]

It follows that \( f \) is compatible with \( g \) implies that \( g \) is compatible with \( f \). Interchanging \( g \) and \( f \) above gives the converse.

It also follows from the above argument that if \( f \) and \( g \) are \( w \)-continuous and \((X, d)\) is a metric space, then the two definitions of compatibility are equivalent.

**Remark 2.** If \((X, t)\) is a \( d \)-complete topological space, \( g \) is \( w \)-continuous, and \( f \) and \( g \) commute, then \( g \) is compatible with \( f \) using definition 2. We use definition 2 for \( d \)-complete topological spaces.

Theorem 3 and its corollaries are generalizations of theorems due to Hicks and Rhoades [4] which are generalizations of theorems due to Jungck [5].

**Theorem 3.** Let \((X, t)\) be a Hausdorff \( d \)-complete topological space and suppose \( f: C \to X \) where \( f \) is \( w \)-continuous and \( C \) is a closed subset of \( X \). Then \( f \) has a fixed point in \( C \) if and only if there exists \( \alpha \in (0, 1) \) and a \( w \)-continuous function \( g: C \to C \) such that \( g(C) \subseteq f(C) \), \( g \) is compatible with \( f \) on \( f^{-1}(C) \) and

\[
(1) \quad d(g(x), g(y)) \leq \alpha d(f(x), f(y)) \quad \text{for all} \ x, y \in C.
\]

Indeed, if \( (1) \) holds, \( f \) and \( g \) have a unique common fixed point.

**Proof.** If \( f(a) = a \) for some \( a \in C \), set \( g(x) = a \) for every \( x \in C \). If \( x \in f^{-1}(C) \), \( f(x) \in C \) and \( g(f(z)) = a \) gives \( g(C) \subseteq f(C) \). Also,

\[
0 = d(a, a) = d(g(x), g(y)) \leq \alpha d(f(x), f(y)) \quad \text{for all} \ x, y \in C.
\]

Suppose there exists \( \alpha \in (0, 1) \) and a \( w \)-continuous function \( g: C \to C \) such that \( g(C) \subseteq f(C) \), \( g \) is compatible with \( f \) on \( f^{-1}(C) \) and \( d(g(x), g(y)) \leq \alpha d(f(x), f(y)) \) for all \( x, y \in C \). Let \( x_0 \in C \).

\[
g(x_0) = f(x_1) \quad \text{for some} \ x_1 \in C \quad \text{since} \ g(C) \subseteq f(C).
\]

Construct a sequence \( \{x_n\} \) with \( \{x_n\} \subseteq C \) and \( f(x_n) = g(x_{n-1}) \) for \( n = 1, 2, 3, \ldots \).

Since

\[
d(f(x_n), f(x_{n+1})) = d(g(x_{n-1}), g(x_n)) \leq \alpha d(f(x_{n-1}), f(x_n)),
\]

it follows that \( d(f(x_n), f(x_{n+1})) \leq \alpha^{n-1} d(f(x_1), f(x_2)) \) for all \( n \). Hence \( \lim_{n \to \infty} f(x_n) = p \) for all \( n \). Then \( f(y_n) \to f(p) \) as \( n \to \infty \), \( g(y_n) \to g(p) \) as \( n \to \infty \), and compatibility give \( f(g(p)) = f(g(y_n)) \to f(p) \) as \( n \to \infty \). Thus, \( f(g(p)) = f(g(p)) \).

Therefore,

\[
(2) \quad d(g(p), g(g(p))) \leq \alpha d(f(p), f(gp)) = \alpha d(g(p), g(gp)) \quad \text{implies} \quad g(p) = g(g(p)).
\]

Hence \( g(p) = g(g(p)) = f(g(p)) \) and \( g(p) \) is a common fixed point of \( f \) and \( g \).

If \( x = f(x) = g(x) \), then \( d(x, g(p)) = d(g(x), g(gp)) \leq \alpha d(f(x), f(gp)) = \alpha d(x, g(p)) \) gives \( x = g(p) \).

**Corollary 1.** Let \((X, t)\) be a Hausdorff \( d \)-complete topological space and \( C \) be a closed
subset of $X$. Suppose $f: C \to X$ and $g: C \to C$, where $f$ and $g$ are $w$-continuous, commute on $f^{-1}(C)$, and $g(C) \subseteq f(C)$. If there exists $\alpha \in (0,1)$ and a positive integer $k$ such that $d(g^k(x), g^k(y)) \leq \alpha d(f(x), f(y))$ for all $x, y \in C$, then $f$ and $g$ have a unique common fixed point.

PROOF. Clearly, $g^k$ commutes with $f$ on $f^{-1}(C)$ and $g^k(C) \subseteq g(C) \subseteq f(C)$. Applying the theorem to $g^k$ and $f$ gives a unique $p \in C$ such that $p = g^k(p) = f(p)$. Since $f$ and $g$ commute on $f^{-1}(C)$ and $p \in f^{-1}(C)$, $g(p) = g(f(p)) = f(g(p)) = g^k(g(p))$ or $g(p)$ is a common fixed point of $f$ and $g^k$. Uniqueness of the common fixed point of $f$ and $g^k$ gives $g(p) = p = f(p)$. If $q = g(q) = f(q)$ then $g^k(q) = f(q)$ and $q = p$.

COROLLARY 2. Let $n$ be a positive integer and let $\alpha > 1$. Suppose $C$ is a closed subset of a Hausdorff $d$-complete topological space and $f: C \to X$ with $C \subseteq f(C)$. If $d(f^n(x), f^n(y)) \geq \alpha d(x, y)$ for all $x, y \in (f^{-1})^{-1}(C)$, then $f$ has a fixed point in $C$.

PROOF. For $n = 1$, this follows from corollary 1 by letting $g = I$. $f^n$ is one-to-one. $C \subseteq f(C)$ implies $C \subseteq f^n(C)$. Let $h$ be the restriction of $(f^n)^{-1}$ to $C$. $h: C \to C$ and $d(h(x), h(y)) \leq \frac{1}{\alpha} d(x, y)$ for all $x, y \in C$. From corollary 1 with $k = 1, h = g^k = g$ and $f = I$, there exists a unique $x_0$ such that $h(x_0) = x_0$. Hence $f(x_0) = f^{n+1}(x_0) = f^n(f(x_0))$ or $h(f(x_0)) = (f^n)^{-1}(f(x_0)) = f(x_0)$. Uniqueness of the fixed point for $h$ gives $x_0 = f(x_0)$. If $f(y) = y$ then $f^n(y) = y$ and $y = h(y)$. Again, uniqueness of the fixed point for $h$ gives $x_0 = y$.

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