ESTIMATES FOR THE CAUCHY MATRIX OF PERTURBED LINEAR IMPULSIVE EQUATION

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ABSTRACT. Estimates for the Cauchy matrix of a perturbed linear impulsive equation are obtained for given estimates for the Cauchy matrix of the corresponding unperturbed linear impulsive equation.

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1. INTRODUCTION.

Consider the linear impulsive equation

\[ \begin{align*}
    z' &= A(t)z, & t &\neq \tau_k, \\
    \Delta z &= A_k z, & t &= \tau_k,
\end{align*} \]

(1.1)

where \( t \) belongs to the interval \( J \subset \mathbb{R}: \tau_k < \tau_{k+1} \) \((k \in \mathbb{Z})\); the sequence \( \{\tau_k\} \) has no finite accumulation point; \( z \in \mathbb{R}^n, A_k \in \mathbb{R}^{n \times n} \). Suppose that \( A(t) \) belongs to the space \( PC(J, \mathbb{R}^{n \times n}) \), i.e. \( A(t) \) is \( n \times n \) matrix-valued function which is continuous for \( t \in J, t \neq \tau_k \), and at the points \( \tau_k \in J \) it has discontinuities of the first kind and is continuous from the left. We recall [1] that the solution \( z(t) \) of (1.1) for \( t \in J, t \neq \tau_k \) satisfies the equation

\[ z' = A(t)z \quad \text{and for } t = \tau_k \]

the conditions

\[ z(\tau_k^-) \overset{\text{def}}{=} \lim_{t \to \tau_k^-} z(t) = z(\tau_k), \quad z(\tau_k^+) \overset{\text{def}}{=} \lim_{t \to \tau_k^+} z(t) = z(\tau_k) + \Delta z(\tau_k) = z(\tau_k) + A_k z(\tau_k). \]

Let \( |x| \) be a norm of the vector \( x \in \mathbb{R}^n \) and \( |A| = \sup \{ |Ax| : |x| = 1 \} \) be the corresponding norm of the matrix \( A \in \mathbb{R}^{n \times n} \). Let the Cauchy matrix \( W(t,s) \) of (1.1) satisfy an estimate of the form

\[ |W(t,s)| \leq \varphi(t)\psi(s) \quad (s, t \in J, s \leq t), \]

(1.2)

where the functions \( \varphi, \psi: J \to \mathbb{R}_+ \) continuous and positive.

Based on estimate (1.2), we shall seek for various estimates for the Cauchy matrix \( Q(t,s) \) of the perturbed linear equation

\[ \begin{align*}
    y' &= [A(t) + B(t)]y, & t &\neq \tau_k, \\
    \Delta y &= [A_k + B_k]y, & t &= \tau_k,
\end{align*} \]

(1.3)
where \( B(t) \in PC(J, \mathbb{R}^{n \times n}) \) and \( B_k \in \mathbb{R}^{n \times n} \).

We shall use the following lemma:

**LEMMA 1.1 [2].** Let the function \( u \in PC(J, \mathbb{R}_+) \) satisfy the inequality

\[
u(t) \leq c + \int_s^t p(\tau)u(\tau)d\tau + \sum_{s \leq \tau_k < t} p_k u(\tau_k) \quad (s, t \in J, s \leq t),\]

where \( c \geq 0 \) and \( p_k \geq 0 \) are constants and \( p(\tau) \in PC(J, \mathbb{R}_+) \).

Then

\[
u(t) \leq c \prod_{s \leq \tau_k < t} (1 + p_k) |\exp \left[ \int_s^t p(\tau)d\tau \right] | \quad (s, t \in J, s \leq t).
\]

2. **MAIN RESULTS.**

Recall [1] that if \( U_k(t, s) \) is the Cauchy matrix for the equation

\[x' = A(t)x \quad (\tau_{k-1} < t \leq \tau_k),\]

then the Cauchy matrix for equation (1.1) is

\[
W(t, s) = U_{k+1}(t, \tau_k^+)\left( I + A_k \right)U_k(\tau_k, s) \quad (\tau_{k-1} < s \leq \tau_k < t \leq \tau_{k+1}),
\]

\[
U_{k+1}(t, \tau_k^+) = \prod_{j=k}^{k+1} \left( I + A_j \right)U_j(\tau_j, \tau_{j-1}^+)(I + A_j)U_j(\tau_j, s) \quad (\tau_{k-1} < s \leq \tau_i < \tau_k < t \leq \tau_{k+1}).
\]

Then an arbitrary solution \( y(t) \) of (1.3) satisfies the integro-summary equation

\[
y(t) = W(t, s)y(s) + \int_s^t W(t, \tau)B(\tau)y(\tau)d\tau + \sum_{s \leq \tau_k < t} W(t, \tau_k^+)B_k y(\tau_k). \quad (2.1)
\]

From (2.1) and (1.2) it follows that

\[
|y(t)| \leq \tilde{\Phi}(t)\psi(s)|y(s)| + \int_s^t \varphi(\tau)\varphi(\tau)|B(\tau)||y(\tau)|d\tau + \sum_{s \leq \tau_k < t} \varphi(\tau_k)\psi(\tau_k)|B_k||y(\tau_k)|.
\]

We apply Lemma 1.1 and obtain the estimate

\[
|y(t)| \leq |y(s)| M(t, s), \quad (2.2)
\]

where

\[
M(t, s) = \tilde{\Phi}(t)\psi(s) \prod_{s \leq \tau_k < t} (1 + \varphi(\tau_k)\psi(\tau_k)|B_k||y(\tau_k)|\exp \left[ \int_s^t \varphi(\tau)\psi(\tau)|B(\tau)|d\tau \right]. \quad (2.3)
\]

From (2.2) and the equality \( y(t) = Q(t, s)y(s) \) there follow immediately the subsequent assertions:

**THEOREM 2.1.** Let the Cauchy matrix \( W(t, s) \) of equation (1.1) satisfy estimate (1.2).

Then the Cauchy matrix \( Q(t, s) \) of equation (1.3) satisfies the estimate
where $M(t,s)$ is given by (2.3)

**COROLLARY 2.1.** If

$$|Q(t,s)| \leq Ke^{\alpha(t-s)} (s,t \in J, s \leq t),$$

where $K > 1$ and $\alpha$ are constants, then

$$|Q(t,s)| \leq Ke^{\alpha(t-s)} \prod_{s \leq \tau_k < t} (1 + K |B_k|) \exp \left[ \int_s^t K |B(\tau)| d\tau \right] (s,t \in J, s \leq t).$$

**COROLLARY 2.2.** If in the interval $J = R_+$ estimate (2.4) is valid and there exists a constant $\delta > 0$ such that

$$s < \tau_k < t$$

and

$$|Q(t,s)| \leq Ke^{\alpha(t-s)} e^{K\delta(t-s) + \ln(1 + K\delta)i[s,t]},$$

where $i[s,t]$ is the number of points $\tau_k$ lying in the interval $[s,t]$.

Moreover, if there exist constants $q \geq 0$ and $\epsilon > 0$ such that

$$i[s,t] \leq q(t-s) + \epsilon,$$

then

$$|Q(t,s)| \leq K(1 + K\delta)^\epsilon \exp[(\alpha + K\delta + q\ln(1 + K\delta))(t-s)] (0 \leq s \leq t).$$

Taking into account that

$$\prod_{s \leq \tau_k < t} (1 + K |B_k|) \leq \exp \left[ \sum_{s \leq \tau_k < t} K |B_k| \right],$$

we obtain

**COROLLARY 2.3.** In the interval $J = R_+$ let estimate (2.4) be valid and let a constant $M > 0$ exist such that

$$\int_0^\infty |B(\tau)| d\tau + \sum_{\tau_k \geq 0} |B_k| \leq M.$$

Then

$$|Q(t,s)| \leq Ke^{KM} e^{\alpha(t-s)} (0 \leq s \leq t).$$

**REMARK 1.** If equation (1.1) is uniformly asymptotically stable, i.e., estimate (2.4) is valid with $\alpha < 0$, then under perturbations for which (2.6) is satisfied with small enough equation (1.3) is also uniformly asymptotically stable.

If equation (1.1) is uniformly stable, i.e., $\alpha = 0$ in (2.4) and condition (2.10) is valid, then equation (1.3) is also uniformly stable.

The goal of the following considerations is to obtain estimates for $Q(t,s)$ in which instead of the integral and the sum of the norms of $B(\tau)$ and $B_k$ the norm of the following function should enter

$$D(s) = \int_s^t B(\tau) d\tau + \sum_{s \leq \tau_k < t} B_k \quad (s,t \in J, s \leq t).$$

We shall note that $D(s)$ is continuous for $s \neq \tau_k$, $D(t^-) = 0$ and $D(\tau_k^-) = D(\tau_k) = D(\tau_k^+) + B_k$.

Let $y(t)$ be an arbitrary solution of (1.3). From (2.1), taking into account that
\[ W(t, t^-) - W(t, s) = \int_s^t \frac{\partial W}{\partial t}(t, r)D(r)y(r)dr + \int_s^t W(t, s)D'(r)y(r)dr + \int_s^t W(t, s)D(r)y'(r)dr \]
\[ + \sum_{s \leq \tau_k < t} [W(t, \tau_k^+)D(\tau_k^+)y(\tau_k^+) - W(t, \tau_k^-)D(\tau_k^-)y(\tau_k^-)]; \]
\[ \frac{\partial W}{\partial t}(t, \tau) = -W(t, \tau)A(\tau), \quad y'(\tau) = [A(\tau) + B(\tau)]y(\tau) \]
and
\[ W(t, \tau_k^+)D(\tau_k^+)y(\tau_k^+) - W(t, \tau_k^-)D(\tau_k^-)y(\tau_k^-) + W(t, \tau_k^-)B_kY(\tau_k) \]
\[ = W(t, \tau_k^+)[D(\tau_k^+)(E + A_k + B_k) - (E + A_k)D(\tau_k^-) + B_k]y(\tau_k) \]
\[ = W(t, \tau_k^+)[D(\tau_k^+)(A_k + B_k) - A_kD(\tau_k^-)]y(\tau_k) \]
we obtain that
\[ y(t) = W(t, s)[E + D(s)]y(s) + \int_s^t W(t, s)[D(\tau)(A(\tau) + B(\tau)) - A(\tau)D(\tau)]y(\tau)dr \]
\[ + \sum_{s \leq \tau_k < t} W(t, s)[D(\tau_k^+)(A_k + B_k) - A_kD(\tau_k^-)]y(\tau_k). \quad (2.12) \]
If \( W(t, s) \) satisfies estimate (1.2) and there exist constants \( M > 0, m > 0 \) and \( \eta > 0 \) such that
\[ |A(t)| \leq M, \; |B(t)| \leq M, \; |A_k| \leq m, \; |B_k| \leq m \]
\[ (t, \tau_k \in J) \quad (2.13) \]
and
\[ \left| \int_s^t B(\tau)dr + \sum_{s \leq \tau_k < t} B_k \right| \leq \eta \quad (s \leq t), \quad (2.14) \]
then from (2.12) we obtain that
\[ |y(t)| \leq \varphi(t)\psi(s)(1 + \eta) |y(s)| + \int_s^t \varphi(t)\psi(\tau) \cdot 3M\eta |y(\tau)| d\tau + \sum_{s \leq \tau_k < t} \varphi(t)\psi(\tau_k) \cdot 3mn |y(\tau_k)| \]
and by Lemma 1.1 we obtain that
\[ |y(t)| \leq |y(s)| N(t, s) \quad (s, t \in J, s \leq t), \quad (2.15) \]
where
\[ N(t, s) = (1 + \eta)\varphi(t)\psi(s) \prod_{s \leq \tau_k < t} (1 + 3M\eta\varphi(\tau_k)\psi(\tau_k)) \exp \left( \int_s^t 3M\eta\varphi(\tau)\psi(\tau) d\tau \right). \quad (2.16) \]
From the estimate (2.15) obtained there follows immediately.
THEOREM 2.2. Let the Cauchy matrix \( W(t, s) \) of equation (1.1) satisfy estimate (1.2) and let conditions (2.13) and (2.14) hold.

Then the Cauchy matrix \( Q(t, s) \) of equation (1.3) satisfies the estimate
\[ |Q(t, s)| \leq N(t, s) \quad (s, t \in J, s \leq t), \]
where \( N(t, s) \) is given by (2.16).
COROLLARY 2.4. If \( |W(t, s)| \leq Ke^{\alpha(t-s)} \quad (s, t \in J, s \leq t) \), then
\[ |Q(t, s)| \leq (1 + \eta)Ke^{\alpha(t-s)} \cdot e^{3KM\eta(t-s)} + \ln(1 + 3Kmn)|s, t| \]
for \( s, t \in J, s \leq t \).
Moreover, if condition (2.8) holds, then

\[ |Q(t, s)| \leq (1 + \eta)(1 + 3K\eta)n e^{(\alpha + 3K\eta + q\ln(1 + 3K\eta))(t - s)} \]  

(2.18)

for \( s, t \in J, s \leq t \).

**COROLLARY 2.5.** In the assumptions of Theorem 2.2 let condition (2.14) be replaced by the more general condition

\[ \int_s^t B(r)dr + \sum_{s \leq \tau_k < t} B_k \leq \eta \quad (s, t \in J, s \leq t \leq s + h), \]  

(2.19)

where \( h > 0 \) is a constant. Then \( Q(t, s) \) satisfies the estimate

\[ |Q(t, s)| \leq K(1 + \eta) e^{[\alpha + 3K\eta + \frac{1}{h}\ln(K + K\eta)](t - s) + \ln(1 + 3K\eta)i[s, t]} \]  

(2.20)

for \( s, t \in J, s \leq t \).

Indeed, estimate (2.20) follows immediately from (2.17) and the fact that the estimate

\[ |y(t)| \leq |y(s)| L e^{\gamma(t - s) + \ln[i[s, t]} \quad (s \leq t \leq s + h) \]

implies

\[ |y(t)| \leq |y(s)| L e^{\gamma + \frac{1}{h}\ln L)(t - s) + \ln[i[s, t]} \quad (s \leq t). \]

**REMARK 2.** In some cases estimate (2.17) is better than estimate (2.7).

**EXAMPLE 1.** Let equations (1.1) and (1.3) be scalar and \( A(t) = -1, \)

\[ B(t) = \sin wt, \quad A_k = 1, \quad B_k = (-1)^k b, \quad 0 \leq b \leq 1, \quad \tau_k = \kappa = 0, 1, 2, \ldots, t \in \mathbb{R}_+. \]

Then

\[ |W(t, s)| = e^{-(t - s) + \ln 2[i[s, t}] \leq Ke^{[t - s]} \quad (0 \leq s \leq t), \]

where \( K = 2, \alpha = 1 + \ln 2. \) In the notation introduced

\[ \delta = 1, \quad M = 1, \quad m = 1, \quad \int_s^t B(r)dr + \sum_{s \leq \tau_k < t} B_k \leq \frac{2}{\delta} + b = \eta. \]

Then \( Q(t, s) \) is estimated:

(i) by estimate (2.7)

\[ |Q(t, s)| \leq Ke^{[t - s]} . e^{2(t - s) + \ln(1 + 2)i[s, t]} \]  

(2.21)

(ii) by estimate (2.17)

\[ |Q(t, s)| \leq (1 + \eta)Ke^{[t - s]} . e^{6\eta(t - s) + \ln(1 + 6\eta)i[s, t]} \]  

(2.22)

Estimate (2.22) is better than estimate (2.21) if \( 6\eta < 2, \) i.e., if \( \frac{2}{\delta} + b < \frac{1}{3} \) which is fulfilled for large \( \omega \) and small \( b. \)

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REFERENCES
