ON RINGS WITH PRIME CENTERS

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ABSTRACT. Let $R$ be a ring, and let $C$ denote the center of $R$. $R$ is said to have a prime center if whenever $ab \in C$ then $a \in C$ or $b \in C$. The structure of certain classes of these rings is studied, along with the relation of the notion of prime centers to commutativity. An example of a non-commutative ring with a prime center is given.

KEY WORDS AND PHRASES. Commutativity, prime center, periodic ring, prime ring.

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1. INTRODUCTION.
Throughout, $R$ is an associative ring, $N$ denotes the set of nilpotent elements of $R$, $C$ denotes the center of $R$, and $J$ denotes the Jacobson radical of $R$.

We introduce the notions of "prime centers" and "semiprime centers" as follows:

DEFINITIONS: A ring $R$ is said to have a prime center if whenever $ab \in C$ then $a \in C$ or $b \in C$. The structure of certain classes of these rings is studied, along with the relation of the notion of prime centers to commutativity. An example of a non-commutative ring with a prime center is given.

BASIC LEMMA 2.
(a) Let $R$ be a ring having a semiprime center. Then $N \subseteq C$ and the idempotent elements of $R$ also belong to the center.
(b) Let $R$ be a ring with identity 1 and having a prime center. Then the units of $R$ are central and hence $J \subseteq C$.

(c) Let $R$ be a prime ring having a semiprime center. If $e$ is an idempotent of $R$, then $e = 0$ or $e = 1$ (if $R$ has 1).

(d) Let $\{R_i, i \in \Gamma\}$ be a family of rings. If the direct sum $\bigoplus_{i \in \Gamma} R_i$ has a prime center, then $R_i$ has a prime center for each $i \in \Gamma$. Moreover, at most one $R_i$ is noncommutative.

**PROOF.** (a) Let $a$ be any nilpotent element in $N$. Then $a^k = 0$ for some positive integer $k$, and hence $a^k \in C$. This implies that $a \in C$ since $R$ has a semiprime center. So $N \subseteq C$. Let $e$ be any idempotent element and $z$ any element of $R$. Then, $(ez - eze) N$ and hence

$$e(x - exe) = (ex - exe)e.$$ 

This implies that $ex = exe$. Similarly $xe = exe$, and hence $e$ belongs to the center of $R$.

(b) Let $u$ be any unit in $R$. Then $uu^{-1} = 1 \in C$. This implies that $u \in C$ or $u^{-1} \in C$, and hence $u \in C$. If $a \in J$ then $a + 1$ is a unit, and hence central. So $J \subseteq C$.

(c) Let $e$ be an idempotent element of $R$. By Lemma 2(a), $e$ is a central idempotent and hence

$$eR(ex - x) = 0 \text{ for all } x \in R.$$ 

This implies, since $R$ is a prime ring, that

$$e = 0 \text{ or } ex = x \text{ for all } x \in R.$$ 

If $e \neq 0$, then $ex = x$ for all $x \in R$. Since $e \in C$, therefore $xe = x$ for all $x \in R$ also. Hence $e = 1$.

(d) Let $C$ be the center of $\bigoplus_{i \in \Gamma} R_i$ and $C_i$ be the center of $R_i$ for each $i \in \Gamma$. Let $a_j$ and $b_j$ be two elements of $R_j(j \in \Gamma)$ such that $a_j b_j \in C_j$. Let $\{a_i\}$ and $\{b_i\}$ be two elements in the direct sum $\bigoplus_{i \in \Gamma} R_i$ such that

$$a_i = \begin{cases} 0 & \text{if } i \neq j \\ a_j & \text{if } i = j \end{cases} \quad b_i = \begin{cases} 0 & \text{if } i \neq j \\ b_j & \text{if } i = j \end{cases}$$ 

Then

$$\{a_i\} \cdot \{b_i\} = \begin{cases} 0 & \text{if } i \neq j \\ a_i b_j & \text{if } i = j \end{cases}.$$ 

This implies that $\{a_i\} \{b_i\} \in C$ since $a_j b_j \in C_j$ (the center of $R_j$). Hence $\{a_i\} \in C$ or $\{b_i\} \in C$ since $\bigoplus_{i \in \Gamma} R_i$ has a prime center. Therefore, $a_j \in C_j$ or $b_j \in C_j$. So $R_j$ has a prime center for each $j \in \Gamma$.

To complete the proof, suppose $R_i$ and $R_j$ are both noncommutative, $i \neq j$. Let $a_i \in R_i \setminus C_i, a_j \in R_j \setminus C_j$ (where $C_i, C_j$ denote the centers of $R_i, R_j$, respectively). Let

$$a_k = \begin{cases} 0 & \text{if } k \neq i \\ a_i & \text{if } k = i \end{cases} \quad b_k = \begin{cases} 0 & \text{if } k \neq j \\ a_j & \text{if } k = j \end{cases}.$$ 

Then $\{a_k\} \{b_k\} = 0 \in C$ while $\{a_k\} \notin C, \{b_k\} \notin C$, contradiction.

We now proceed to prove the main theorems.

**THEOREM 1.** Let $R$ be a periodic ring. Then $R$ is commutative if and only if $R$ has a semiprime center.

**PROOF.** If $R$ is commutative, then $R$ clearly has a semiprime center. Suppose that $R$ has a semiprime center and let $x$ be any element of $R$. Since $R$ is periodic, Lemma 1 implies that $x^k$
is idempotent for some positive integer $k$. Hence by Lemma 2(a), $x^k \in C$. This implies that $x \in C$, since $R$ has a semiprime center.

**Theorem 2.** Let $R$ be a prime ring with a semiprime center. Then $R$ is a domain.

**Proof.** By Lemma 2(a), $N \subseteq C$. But the center of a prime ring has no nonzero zero divisors, and hence $N = \{0\}$. It is well known that a prime ring with $N = \{0\}$ is a domain.

**Theorem 3.** Let $R$ be an Artinian ring with prime center, and $N$ the set of nilpotent elements. Then $N$ is an ideal and $R/N$ is a direct sum of fields.

**Proof.** By Lemma 2(a), $N \subseteq C$; and this implies that $N$ is an ideal. Now $R/N$ is Artinian and reduced, so by the Wedderburn-Artin structure theory, $R/N$ is a (finite) direct sum of division rings. Let $R_1$ be a division ring which is an internal direct summand of $R/N$, and let $x + N$ be the identity element of $R_1$. Since idempotents may be lifted, there exists an idempotent $e \in R$ such that $e + N$. By Lemma 2(a), $e \in C$.

**Corollary 1.** Let $R$ be a prime Artinian ring with identity $1$. If $R$ has a prime center, then $R$ is a field.

**Proof.** Let $I$ be a nonzero ideal of $R$. Then $I$ is non-nilpotent, since $R$ is a prime ring. This implies that $I$ contains a nonzero idempotent element, since $R$ is Artinian ([2], Theorem 1.3.2). So, by Lemma 2(c), $1 \in I$ and hence $I = R$. This implies that $R$ is a simple ring. Hence, by Theorem 3, $R$ is a field.

**Corollary 2.** Let $R$ be a semisimple Artinian ring. If $R$ is a prime center, then $R$ is isomorphic to a direct sum of fields.

**Proof.** This follows at once from Theorem 3, since $N = \{0\}$.

**Theorem 4.** Let $R$ be a ring with identity and having a prime center. If for each $x \in R$, there is a monic polynomial $f = f_x$ with integer coefficients such that $f(x) \in C$, then $R$ is commutative.

**Proof.** We will prove that if $x \in R$ with $f_x(x) \in C$, then $x \in C$. We proceed by induction on the degree $n$ of the polynomial $f_x$. If $n = 1$, then $x \in R$ with $f_x(x) = x + a_0 \in C$. This implies that $x \in C$. Suppose that the above statement is true for all $x \in R$ having a polynomial $f_x$ of degree $n$ with $f_x(x) \in C$. Let $y \in R$ with

$$f_y(y) = y^{n+1} + a_n y^n + \cdots + a_1 y + a_0 \in C$$

where $a_0, \ldots, a_n$ are integers. Thus

$$y^{n+1} + a_n y^n + \cdots + a_1 y \in C,$$

and hence

$$y(y^n + a_n y^{n-1} + \cdots + a_2 y + a_1) \in C.$$  

If $y \notin C$, then $y^n + a_n y^{n-1} + \cdots + a_1 \in C$ since $R$ has a prime center. So, $y$ has a polynomial $g$ of degree $n$ with $g(y) \in C$. Hence $y \in C$ by the induction hypothesis. This completes the proof of Theorem 4.

**3. Example of a Noncommutative Ring with a Prime Center.**

Choose $F$ to be an infinite field admitting an automorphism $\sigma$ of infinite order. Let $F[x, \sigma]$
be the skew polynomial ring of all polynomials $p(x)$ over $F$ such that $x^n a = \sigma^n(a) x^n$ for every $a \in F$ and every positive integer $n$. It is easy to verify that $F[x, \sigma]$ is a domain.

Let $R = F[x, \sigma]$. We will show that $R$ is a noncommutative ring having a prime center. Let $C$ denote the center of $R$ and let $P(x) = a_1 x + a_2 x^2 + \cdots + a_n x^n$ be any nonzero element in the center of $R$, it being assumed that $a_n \neq 0$. Now,

$$xP(x) = xa_1 x + xa_2 x^2 + \cdots + xa_n x^n$$

and

$$P(x)x = a_1 x^2 + a_2 x^3 + \cdots + a_n x^{n+1}.$$

But $xP(x) = P(x)x$, since $P(x) \in C$. This implies that

$$\sigma(a_1) = a_1, \sigma(a_2) = a_2, \ldots, \sigma(a_n) = a_n. \quad (3.1)$$

Since $\sigma$ has an infinite order, there exists an element $a \in F$ such that

$$\sigma^n(a) \neq a. \quad (3.2)$$

Now,

$$axP(x) = axa_1 x + axa_2 x^2 + \cdots + axa_n x^n$$

$$= ao(a_1)x^2 + ao(a_2)x^3 + \cdots + ao(a_n)x^{n+1}$$

$$= a_1 o x^2 + a_2 o x^3 + \cdots + a_n o x^{n+1} \quad \text{by (3.1)}$$

Also,

$$P(x)ax = a_1 x a x + a_2 x a x^2 + \cdots + a_n x a x^n$$

$$= a_1 o x^2 + a_2 o x^3 + \cdots + a_n o x^{n+1}.$$

But $axP(x) = P(x)ax$, since $P(x) \in C$. This implies that

$$aa_1 = a_1 o(a), aa_2 = a_2 o(a), \ldots, aa_n = a_n o(a). \quad (3.3)$$

Since $F$ is a field and $a_n \neq 0$, the last of these equations contradicts (3.2). Hence $P(x) = 0$, and the center $C = \{0\}$. Thus, $R = F[x, \sigma]$ is a domain with center $\{0\}$. This implies that $R$ has a prime center. For, if $P(x)q(x) \in C = \{0\}$, then $P(x) \cdot q(x) = 0$. This implies that $P(x) = 0 \in C$ or $q(x) = 0 \in C$, since $R$ is a domain. $R$ is clearly a noncommutative ring ($xa = \sigma(a)x \neq ax$).

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