EXISTENCE OF WEAK SOLUTIONS FOR ABSTRACT HYPERBOLIC-PARABOLIC EQUATIONS

MARCONDES RODRIGUES CLARK
Universidade Federal da Paraíba - Campus II - DME - CCT
58.109-970 - Campina Grande - Paraíba - Brasil
e-mail: mclark at brufpb2.bitnet

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ABSTRACT
In this paper we study the Existence and Uniqueness of solutions for the following Cauchy problem:

$$A_2u''(t) + A_1u'(t) + A(t)u(t) + M(u(t)) f(t), \quad t \in (0, T)$$

$$u(0) = u_0; A_2u'(0) = A_2^{1/2}u_1;$$

where $A_1$ and $A_2$ are bounded linear operators in a Hilbert space $H$, $\{A(t)\}_{0 \leq t \leq T}$ is a family of self-adjoint operators, $M$ is a non-linear map on $H$ and $f$ is a function from $(0, T)$ with values in $H$.

As an application of problem (1) we consider the following Cauchy problem:

$$k_2(x)u'' + k_1(x)u' + A(t)u + u^3 f(t) \text{ in } Q,$$

$$u(0) = u_0; k_2(x)u'(0) = k_2(x)^{1/2}u_1$$

where $Q$ is a cylindrical domain in $\mathbb{R}^d$; $k_1$ and $k_2$ are bounded functions defined in an open bounded set $\Omega \subset \mathbb{R}^d$,

$$A(t) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a_{ij}(x,t)) \frac{\partial}{\partial x_i};$$

where $a_{ij}$ and $a'_{ij} = \frac{\partial}{\partial t}u_{ij}$ are bounded functions on $\Omega$ and $f$ is a function from $(0, T)$ with values in $L^2(\Omega)$.


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INTRODUCTION
Let $T > 0$ be a positive real number and $\Omega$ be a bounded open set of $\mathbb{R}^n$, with smooth boundary $\Gamma$. In the cylinder $Q = \Omega \times (0, T)$, Bensoussan et al. [01], studied the homogeneization for the following Cauchy problem:

$$k_2(x)u'' + k_1(x)u' - \Delta u = f \quad \text{in } Q,$$

$$u(x, 0) = u_0(x), \quad k_2(x)u'(x, 0) = k^{1/2}(x)u_1(x), \quad x \in \Omega$$

Many authors have been investigating the existence of solution for non-linear equations associated with problem (3),

see: Larkin [04], Lima [05], Medeiros [07-09], Melo [10], Maciel [11], Neves [12] and Vagrov [15].

Other interesting results relative to existence of a solution for a non-linear equation associated with the equation of the problem (3) can be found in the work of Jörgens [03]
In this work he proved the existence of classical solution by iterative methods for the mixed problem associated to the equation

\[ u_{tt} - \Delta u + F'(|u|^2)u = 0, \]

in open domain of \(\mathbb{R}^3\), with the hypothesis \(F(0) = 0\) and \(|F'(s)| \leq a[b + F(s)]^{\alpha}\) where \(a, b\) and \(\alpha\) are positive constants with \(\alpha < \frac{3}{2}\).

In Section 1, we establish some notation for the function spaces and conditions for \(A_1, A_2, \{A(t)\}_{0 \leq t \leq T}, M\) and \(f\) in equation (1). In Section 2, we state our main results and we prove the assertions made. In the final Section we make an application of problem (1).

**1. PRELIMINARIES**

We will assume that standard function spaces are known: \(C^k(\Omega), L^p(\Omega), H^k(\Omega), H^k_0(\Omega), C(0, T; X), L^p(0, T; X)\) where \(X\) is a Banach space.

Let \(H\) be a real Hilbert space, with inner product and the norm denoted by \((\cdot, \cdot)\) and \(\|\cdot\|\), respectively.

We consider here the following assumptions:

i) \(A_2 : H \to H\), a positive symmetric operator

ii) \(A_1 : H \to H\), a symmetric operator such that:

\[(A_1u, u) \geq \beta\|u\|^2, \quad 0 < \beta \in \mathbb{R}, \quad \text{for all } u \in H.\]

iii) Let \(\{A(t); \quad 0 \leq t \leq T\}\) be a family of self-adjoint linear operators of \(H\), such that there exists a constant \(\alpha > 0\), satisfying \((A(t)u, u) \geq \alpha\|u\|^2\) for all \(u \in D(A(t))\), where we assume that the domain \(D(A(t))\) of \(A(t)\) is constant, i.e., \(D(A(t)) = D(A(s)) \forall t, s \geq 0\). It is known from the spectral theory for self-adjoint operators that there exists only one positive self-adjoint operator \(A^{\frac{1}{2}}(t)\) such that:

\[D(A(t)) \subseteq D(A^{\frac{1}{2}}(t)).\]

From assumption iii) we have, see Medeiros [09], that \(D(A^{\frac{1}{2}}(t))\) is constant.

Let \(V_t = D(A^{\frac{1}{2}}(t))\) with inner product \((\cdot, \cdot)\) and associated norm \(\|\cdot\|_t\). Therefore \(\|u\|^2_t = |A^{\frac{1}{2}}(t)u|^2 \geq \alpha\|u\|^2\).

So that, \(V_t\) is a Hilbert space, dense and embedded in \(H(V_t \hookrightarrow H)\), and \(V_t\) is isomorphic with \(V_0, \forall t\).

iv) \(A(t)\) is continuously strongly differentiable.

v) For \(u \in D(A(0))\), we assume that there exists a real \(\gamma > 0\), independent from \(t\), such that:

\[ (A'(t)u, u) \leq \gamma\|u\|^2_0, \quad \forall t \in [0, T] \]

vi) We assume that the embedding \(V_0 \hookrightarrow H\) is compact. Therefore, the spectrum of the operator \(A(t)\) is discret.
Identifying \( H \) with his dual \( H' \), we have the immersions:

\[ V_0 \hookrightarrow H \hookrightarrow V_0' \]; where each space is dense on the following one.

In this work, we use the symbol \( \langle \cdot , \cdot \rangle \), to denote the duality between \( V_0' \) and \( V_0 \). Sometimes it means an application of a Vector distribution to a real test function.

vii) Let \( M \) be an operator of \( V_0 \) in \( H \) satisfying the following conditions:

a) \( M \) is monotone, hemi-continuous and bounded (in the sense of taking bounded sets of \( V_0 \) into bounded sets of \( H \)).

b) There exists a constant \( \sigma > 0 \) so that

\[ \int_0^T (M(u(s)), u'(s))ds \geq -\sigma \forall t \in [0, T] \text{ and } \forall u \in E_C \]

where \( E_C \) denotes the set \( \{ u \in L^\infty(0, T; V_0); u' \in L^2(0, T; H) \text{ and } \|u(0)\|_0 \leq C \} \)

2.1 The Main Results

**Theorem - 1:** (Existence) Under the above assumptions (i-vii) and considering

\[ f \in L^2(0, T; H) \quad (2.1) \]

\[ u_0 \in V_0 \quad (2.2) \]

\[ u_1 \in H, \quad (2.3) \]

then there exists a function \( u \) defined in \( (0, T) \) with values in \( V_0 \) such that:

\[ u \in L^\infty(0, T; V_0) \quad (2.4) \]

\[ u' \in L^2(0, T; H), \quad (2.5) \]

besides this, \( u \) is a solution of problem (1) in the following way:

\[ -\int_0^T (A_2u'(t), \Phi'(t)v)dt + \int_0^T (A_1u'(t), \Phi(t)v)dt + \]

\[ + \int_0^T (A_{1/2}(t)u(t), A_{1/2}(t)\Phi(t)v)dt + \]

\[ + \int_0^T (M(u(t)), \Phi(t)v)dt = \int_0^T (f(t), \Phi(t)v)dt, \forall v \in V_0 \]

and \( \forall \Phi \in C^1_0(0, T) \).

\[ u(0) = u_0 \quad (2.7) \]

\[ A_2u'(0) = A_{1/2}u_1. \quad (2.8) \]

For the uniqueness we need the following condition on \( M \):

viii) Given \( C > 0 \), there exists \( K > 0 \), which depends on \( C \), such that:

\[ |M(u) - M(v)| \leq K|u - v| \]

for all \( u, v \in V \) whenever \( \|u\|_0 \leq C \) and \( \|v\|_0 \leq C \).
Theorem - 2. (Uniqueness) Suppose that the operators $A_1, A_2, A(t)$ satisfy the conditions of Theorem-1 and (viii), respectively, and $M$ maps functions of $L^\infty(0, T; V_0)$ into functions of $L^2(0, T; H)$. Then, there exists at most one function $u$ in the class

$$u \in L^\infty(0, T; V_0), u' \in L^2(0, T; H),$$

and $u$ is a solution of problem (1) in the sense (2.6) - (2.8) of Theorem-1.

Remark 2.1

From (2.4), (2.5) and (2.6) we obtain that $A_2u'' \in L^2(0, T; V'_0)$ and this together with (2.4) (2.5) imply that the initial conditions (2.7) (2.8) make sense.

2.2 Proof of the Theorems

In this part we use the followin result:

Lema 1. Let $u \in L^2(0, T; H), u' \in L^2(0, T; V'_0)$ with $v$, and $v' \in L^2(0, T; H)$. Then

$$\frac{d}{dt} <u, v> = <u', v> + (u, v').$$

For the proof of this lemma see Tanabe, [13].

We apply the standard Galerking approximate procedure. Let $(w_v)$ be a base of $D(A(0))$ that it is a base of $H$, by density. From the assumption (i), we have $((A_2 + \lambda I)\frac{1}{2}w_v)$ is also a base of $H$; where $\lambda > 0$ is a constant. Let $V_m(0)$ be a subspace of $D(A(0))$ generated by the first-$m$ vectors $w_1, \ldots, w_m$, and $V'_m(0)$ the subspace generated by first-$m$ vectors $(A_2 + \lambda I)^{\frac{1}{2}}w_1, \ldots, (A_2 + \lambda I)^{\frac{1}{2}}w_m$.

We put $u_{\lambda m}(t) = \sum_{i=1}^{m} g_{\lambda m}(t)w_i$ as a solution of the approximate perturbed problem:

$$((A_2 + \lambda I)u''_{\lambda m}(t) + A_2u_{\lambda m}(t) + A(t)u_{\lambda m}(t) + (M(u_{\lambda m}(t)), v) =$$

$$= (f(t), v), \forall v \in V_m(0).$$

$$u_{\lambda m}(0) = u_{0m}; \text{ where } u_{0m} = \sum_{i=1}^{m} \alpha_{im}w_i \rightarrow u_0$$

(2.9)

strongly in $V_0$

$$u'_{\lambda m}(0) = u_{1\lambda m}; \text{ where } u_{1\lambda m} = \sum_{i=1}^{m} \beta_{i\lambda m}w_i$$

(2.10)

where the coefficient $\beta_{i\lambda m}$ denotes the coordinates of the vector $P_{\lambda m}u_1$, the orthogonal projection of the vector $u_1$ upon the subspace $V_m^\lambda(0)$ in relation to the base $((A_2 + \lambda I)^{\frac{1}{2}}v_v)$, such that:

$$P_{\lambda m}u_1 = \sum_{i=1}^{m} \beta_{i\lambda m}(A_2 + \lambda I)^{\frac{1}{2}}w_i.$$  

(2.11)

We have that $P_{\lambda m}u_1 \rightarrow u_1$ strongly in $H$ and satisfies

$$|P_{\lambda m}u_1| \leq |u_1| \forall m \quad \forall \lambda > 0.$$

System (2.9) - (2.11) is equivalent to a system of non-linear ordinary differential equations, which has a solution $u_{\lambda m}(t)$ by using Caratheodory's theorem, see Coddington - Levinson [02]; defined in an interval $[0, t_m)$, with $t_m < T$, for each $m \in \mathbb{N}$.
2.3 - "A priori" Estimates

In (2.9) taking \( v = 2u'_{\lambda m}(t) \) we have:

\[
\frac{d}{dt}[(A_2 + \lambda I)^{\frac{1}{2}}u'_{\lambda m}(t)]^2 + 2(A_1u'_{\lambda m}(t), u'_{\lambda m}(t)) + \\
2(A_1(t)u_{\lambda m}(t), A_1(t)u'_{\lambda m}(t)) + 2(M(u_{\lambda m}(t)), u'_{\lambda m}(t)) = \\
2(f(t), u'_{\lambda m}(t)).
\]

Using the above assumptions, we have,

\[
\|(A_2 + \lambda I)^{\frac{1}{2}}u'_{\lambda m}(t)\|^2 + \beta \int_0^t |u'_{\lambda m}(s)|^2 \, ds + \|u_{\lambda m}(t)\|^2 \leq \\
2\sigma + |P_{\lambda m}u_1|^2 + \|u_{\text{om}}\|^2 + \int_0^t (A' s)u_{\lambda m}(s), u_{\lambda m}(s) ds + \\
\frac{1}{\beta} \int_0^t |f(s)|^2 \, ds.
\]

From (2.1), (2.10) and (2.11), there exists a constant \( C(\cdot) \) such that

\[
\|(A_2 + \lambda I)^{\frac{1}{2}}u'_{\lambda m}(t)\|^2 + \beta \int_0^t |u'_{\lambda m}(s)|^2 \, ds + \|u_{\lambda m}(t)\|^2 \leq C + \\
\int_0^t (A' s)u_{\lambda m}(s), u_{\lambda m}(s) ds.
\]

(*) Let us denote by \( C \) various constants.

It is not difficult to prove that the function \( g(t) = \|u_{\lambda m}(t)\|^2 \) is continuous. So that from Gronwall's inequality, from \( V_t \equiv V_0 \), and from the assumption (v), we conclude that:

\[
\|u_{\lambda m}(t)\|_0 \leq C \tag{2.12}
\]

independently from \( \lambda > 0 \) \( m \in N \) and of \( t \in [0, t_m) \). So that, we have

\[
\|(A_2 + \lambda I)^{\frac{1}{2}}u'_{\lambda m}(t)\|^2 + \beta \int_0^t |u'_{\lambda m}(s)|^2 \, ds + \|u_{\lambda m}(t)\|^2 \leq C \tag{2.13}
\]

independently from \( \lambda > 0 \) \( m \in N \) and of \( t \in [0, t_m) \).

Therefore, from (2.12), (2.13) and by Carathéodory Theorem there exists a solution in all interval \([0, T]\).

So we obtain the following estimates:

\[
\|u_{\lambda m}\|_{L^\infty(0,T;V_0)} \leq C, \quad \forall \lambda > 0, \ m \in N. \tag{2.14}
\]

\[
\|u'_{\lambda m}\|_{L^2(0,T;H)} \leq C, \quad \forall \lambda > 0, \ m \in N. \tag{2.15}
\]

Where \( C \) is a constant independent of \( m \in N \) and \( \lambda > 0 \). From the estimate (2.14) and noting that \( M \) is bounded it follows that

\[
\|M(u_{\lambda m})\|_{L^\infty(0,T;H)} \leq C, \quad \forall \lambda > 0, \ m \in N. \tag{2.16}
\]

The estimates (2.14) - (2.16), imply that there exists a subsequence of \( (u_{\lambda m}) \), still denoted by \( (u_{\lambda m}) \), and a function \( u_\lambda \) such that

\[
u_{\lambda m} \rightarrow u_\lambda \text{ weak-star in } L^\infty(0,T;V_0). \tag{2.17}
\]

\[
u'_{\lambda m} \rightarrow u'_\lambda \text{ weak in } L^2(0,T;H) \tag{2.18}
\]
\[ A^{1/2}(t)u_{\lambda m} \rightharpoonup A^{1/2}(t)u_{\lambda} \text{ weak-star in } L^{\infty}(0,T;H) \] (2.19)

\[ (A_2 + \lambda I)u_{\lambda m}' \rightharpoonup (A_2 + \lambda I)u_{\lambda}' \text{ weak in } L^2(0,T;H) \] (2.20)

\[ A_1u_{\lambda m}' \rightharpoonup A_1u_{\lambda}' \text{ weak in } L^2(0,T;H) \] (2.21)

\[ M(u_{\lambda m}') \rightharpoonup \chi \text{ weak-star in } L^{\infty}(0,T;H) \] (2.22)

The fact that \( A^{1/2}(t); A_1 \text{ and } A_2 \) are weakly closed operators of \( L^2(0,T;H) \) was used in (2.19), (2.20) and (2.21).

\section*{2.4 - The Nonlinear Term}

Since \( H \rightharpoonup V_0 \) continuously, it follows from (2.15) that:

\[ \|u_{\lambda m}'\|_{L^2(0,T;V_0^*)} \leq C, \text{ independently of } \lambda > 0 \text{ and } m \in \mathbb{N}. \] (2.23)

From (2.4), (2.23) and by the compact embedding from \( V_0 \) in \( H \), it follows from the Lemma of Aubin-Lions, see Lions [06], that:

\[ u_{\lambda m} \rightharpoonup u_{\lambda} \text{ strong in } L^2(0,T;H). \] (2.24)

For \( v \in L^2(0,T;V) \) and \( \Theta > 0 \) a real number, by the monotonicity of \( M \) we have:

\[ \int_0^T (M(u_{\lambda} + \Theta v) - M(u_{\lambda m}), u_{\lambda} + \Theta v - u_{\lambda m}) dt \geq 0. \]

From this inequality, taking the limit \( m \to \infty \) and using the convergences (2.22) and (2.24) we get:

\[ \int_0^T (M(u_{\lambda} + \Theta v) - \chi, v) dt \geq 0, \quad \forall \ v \in L^2(0,T;V). \]

It follows, by the hemicontinuity of \( M \), that:

\[ M(u_{\lambda}) = \chi. \] (2.25)

By multiplying both sides of (2.9) by \( \Phi \in C_0^\infty(0,T) \), integrating from \( t = 0 \) to \( t = T \), passing to the limit and using the convergences (2.19) - (2.22) we obtain,

\[ -\int_0^T ((A_2 + \lambda I)u_{\lambda}', \Phi v) dt + \int_0^T (A_1u_{\lambda}', \Phi v) dt + \int_0^T (A^{1/2}(t)u_{\lambda}, A^{1/2}(t)\Phi v) dt + \int_0^T (M(u_{\lambda}), \Phi v) dt = \]

\[ = \int_0^T (f, \Phi v) dt, \quad \forall \ \Phi \in C_0^\infty(0,T), \quad \forall \ v \in V. \] (2.26)

Since the linear combinations of \( w_1, \ldots, w_m \) are dense in \( D(A(0)) \), it follows that the above equality, remains valid for all \( v \in D(A(0)) \) and for all \( \Phi \in C_0^\infty(0,T) \) also. So that, \( u_{\lambda} \) is a solution of the perturbed problem in the sense given in (2.6).

From this we have that

\[ ((A_2 + \lambda I)u_{\lambda}')' = -A_1u_{\lambda}' - A(t)u_{\lambda} - M(u_{\lambda}) + f \in L^2(0,T;V_0^*). \] (2.27)

Noticing that the estimates (2.14) - (2.16) are independent of \( \lambda > 0 \), we obtain the same convergences (2.17) - (2.22) and also the equality (2.25) replacing \( u_{\lambda m} \) by \( u_{\lambda} \) and \( u_{\lambda m}' \) by \( u_{\lambda}' \).

By the above arguments, taking the limit in (2.26) we have that \( u \) satisfies (2.4)-(2.6).
From (2.6) we have,
\[ (A_2 u')' + A_1 u' + A(t)u + M(u) = f \quad \text{in} \quad L^2(0, T; V'_0). \]  
(2.28)
\[ (A_2 u')' \in L^2(0, T; V'_0). \]  
(2.29)

### 2.5 - The Initial Conditions

The proof of the initial conditions (2.7) and (2.8) are obtained by the convergences (2.17), (2.18). Let \( \Phi \in C^1([0, T]) \) with \( \Phi(0) = 1, \Phi(T) = 0 \), and \( v \in V_0 \). Then by (2.17) and using Lemma 1, with \( u \in V_0 \), we obtain

\[ \langle (A_2 + \lambda I)u'_1(0), v \rangle > - \int_0^T ((A_2 + \lambda I)u'_1, \Phi')dt + \int_0^T (A_1 u'_1, \Phi v)dt \]
\[ + \int_0^T < A(t)u_1, \Phi v > dt + \left\langle (M(u), \Phi v)dt - (f, \Phi v)dt. \right\rangle \]

Taking the limit in the above equality, we obtain

\[ \langle A_2^{\frac{3}{2}} u_1, v \rangle > - \int_0^T (A_2 u', \Phi')dt + \int_0^T (A_1 u', \Phi v)dt \]
\[ + \int_0^T < A(t)u, \Phi v > dt + \int_0^T (M(u), \Phi v)dt = \int_0^T (f, \Phi v)dt. \]  
(2.30)

Integrating by parts \( \int_0^T (A_2 u'_1, \Phi' v)dt \), observing (2.29) and using Lemma 1, we get from (2.28) and (2.30) that:

\[ \langle A_2 u'(0), v \rangle = \langle A_2^{\frac{3}{2}} u_1, v \rangle, \quad \forall \ v \in V. \]

From this it follows the proof of Theorem 1.

**Remark 1.** We obtain the same Theorem 1 by considering:

\[ M : L^2(0, T; V_0) \rightarrow L^2(0, T; H) \]

pseudo-monotone and satisfying condition (vii) (see Lions, [06]).

### 3.- PROOF OF THEOREM 2

If \( u \) and \( v \) satisfy Theorem-1, then \( w = u - v \) satisfies:

\[ (A_2 w')' + A_1 w' + A(t)w + M(u) - M(v) = 0 \quad \text{in} \quad L^2(0, T; V'_0). \]  
(3.1)
\[ w(0) = 0, \quad A_2 w'(0) = 0. \]  
(3.2)

We'll prove that \( w = 0 \) in \([0, T] \).

We observe that the solution \( u'(t) \in H \) and \( (A_2 u')'(t) \in V' \). Therefore it doesn't make sense the duality between these vectors. In this case, we'll use the method introduced by Visik-Ladyzenskaja [14].

For each \( s \) with \( 0 < s < T \), we'll consider the function \( z(t) \) given by:

\[ z(t) = \begin{cases} 
- \int_t^s w(\xi)d\xi & \text{if } 0 \leq t \leq s \\
0 & \text{if } s < t \leq T
\end{cases} \]  
(3.3)

We have that \( z(s) = 0, \quad z'(t) = w(t) \) for \( 0 \leq t \leq s \) and \( z(t) \in V_0 \) for each \( t \in [0, T] \).
Defining \( w_1(t) \) by \( w_1(t) = \int_0^t w(\gamma)d\gamma \), we have \( z(t) = w_1(t) - w_1(s), \quad 0 \leq t \leq s \).

Taking the duality of (3.1) with (3.3) and integrating from \( t = 0 \) to \( t = T \), we obtain
\[
\int_0^T < (A_2w'), z > dt + \int_0^T (A_1w', z)dt + \int_0^T < A(t)w, z > dt + \int_0^T (M(u) - M(v), z)dt = 0.
\] (3.4)

We have that:
\[
\int_0^T < (A_2w'), z > dt = -\frac{1}{2}(A_2w(s), w(s))
\]
\[
\int_0^T (A_1w', z)dt = -\int_0^T (A_1w, w)dt.
\]
\[
\int_0^T A^{\frac{1}{2}}(t)z(t)dt = \frac{1}{2} \int_0^T \frac{d}{dt}||z(t)||^2 dt - \frac{1}{2} \int_0^T (A'(t)z(t), z(t))dt = -\frac{1}{2}||A^{\frac{1}{2}}(0)w_1(s)||^2 - \frac{1}{2} \int_0^T (A'(t)z(t), z(t))dt.
\]

Substituting the above equalities in (3.4) we have:
\[
\frac{1}{2}||A^{\frac{1}{2}}w(s)||^2 + \int_0^T (A_1w, w)dt + \frac{1}{2}||A^{\frac{1}{2}}(0)w_1(s)||^2 = \int_0^T (M(u) - M(v), z)dt - \frac{1}{2} \int_0^T (A'(t)z(t), z(t))dt.
\]

By using hypotheses ii), iii), v), viii) in the above equality, we obtain:
\[
\frac{1}{2}||A^{\frac{1}{2}}w(s)||^2 + \beta \int_0^T |w(t)|^2 dt + \frac{\alpha^2}{2}|w_1(s)|^2 \leq \mu |w(t)||z(t)|dt + \frac{\gamma}{2} \int_0^T |z(t)|^2 dt \leq \mu |w(t)||w_1(t)|dt + \frac{\gamma}{2} \int_0^T |z(t)|^2 dt.
\]

By applying the inequality \( ab \leq \frac{\lambda a^2}{2} + \frac{b^2}{2\lambda}, \quad \forall \lambda > 0 \), in the above inequality one has:
\[
(\beta - \mu^2 \lambda) \int_0^T |w(t)|^2 dt + \left[ \frac{\alpha}{2} - \left( \frac{1}{2\lambda} + \gamma \right)s \right] |w_1(s)|^2 \leq (\frac{\beta}{2\lambda} + \gamma) \int_0^T |w_1(t)|^2 dt, \quad \forall \lambda > 0 \text{ such that } \beta - \mu^2 \lambda > 0
\]

and \( \frac{\alpha}{2} - \left( \frac{1}{2\lambda} + \gamma \right)s > 0 \). If we choose \( \lambda > 0 \) such that \( \beta - \mu^2 \lambda = \frac{\beta}{2} \), that is, \( \lambda = \frac{\beta}{2\mu^2} \) and \( s_0 \) such that \( \frac{\alpha}{2} - \left( \frac{1}{2\lambda} + \gamma \right)s_0 = \frac{\alpha}{4} \), that is, \( s_0 = \frac{\alpha \lambda}{2(1 + 2\lambda \gamma)} \), we obtain from the above equality:
\[
\frac{\beta}{2} \int_0^T |w(t)|^2 dt + \frac{\alpha^2}{4}|w_1(s)|^2 \leq \left( \frac{\mu^2}{\beta} + \gamma \right) \int_0^T |w_1(t)|^2 dt
\] (3.5)

\( \forall s \in [0, s_0] \). Gronwall’s inequality implies that \( w_1(s) = 0 \) for all \( s \in [0, s_0] \). Which implies \( w_1(s) = 0, \quad \forall s \in [0, s_0], \) consequently \( w(t) = 0 \) for all \( t \in [0, s_0] \).

Using the same argument in \([0, s_0]\) for the Cauchy problem:
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\[ \begin{bmatrix} (A_2 w')' + A_1 w' + A(t) w + M(u) - M(v) = 0 \\ w(s_0) = 0, \ A_2 w'(s_0) = 0 \end{bmatrix} \]

we obtain that \( w(t) = 0 \), for all \( t \in [s_0, 2s_0] \).

After a finite number of steps we conclude \( w(t) = 0 \) in \([0, T]\) and the proof of the Theorem 2 is completed.

3. EXAMPLES

1) Let \( \Omega \) be a regular bounded open subset of \( \mathbb{R}^n \) and \( H = L^2(\Omega) \), \( V = H_0^1(\Omega) \).

Let us define the functions \( k_1, k_2 \in L^\infty(\Omega) \) such that \( k_1(x) \geq \beta > 0 \) a.e. and \( k_2(x) \geq 0 \) a.e. in \( \Omega \) where \( \beta \) is a constant.

We define the operators \( A_1 \) and \( A_2 \) in \( L^2(\Omega) \) by

\[ (A_1 u)(x) = k_1(x) u(x), \quad (A_2 u)(x) = k_2(x) u(x) \]

and consider

\[ A(t) = - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a_{ij}(x, t) \frac{\partial}{\partial x_i}) \]

being the domain of \( A(t) \) the space \( H^2(\Omega) \cap H_0^1(\Omega) \) which is dense in \( L^2(\Omega) \); where \( a_{ij} = a_{ji} \) and

\[ a_{ij} = \frac{\partial}{\partial t} a_{ij} \in L^\infty(\Omega \times (0, T)), \; \forall 1 \leq i, j \leq n. \]

Then \( A(t) \) is a family of self-adjoint operators.

We also assume that:

\[ \sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \geq \gamma (|\xi_1|^2 + \ldots + |\xi_n|^2); \]

\( (x, t) \in Q, \; 0 < \gamma \in \mathbb{R} \) and \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \); then, by Poincaré-Friedrichs inequality implies that \( (A(t) u, u) \geq \alpha |u|^2 \), for all \( u \in D = D(A(t)) \) and for some constant \( \alpha > 0 \).

Noting that

\[ |((A(t) - A(t_0)) u, u)| \leq \sum_{i,j=1}^n \int_{\Omega} |a_{ij}(x, t) - a_{ij}(x, t_0)| \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx, \]

being \( a_{ij} \in L^\infty(Q) \), we have that there exists the \( \lim_{t \to t_0} (A(t) u - A(t_0) u) \) in norm of \( L^2(\Omega) \).

Therefore \( A(t) \) is continuously strongly differentiable.

Being \( A'(t) = - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a'_{ij}(x, t) \frac{\partial}{\partial x_i}) \) with \( a'_{ij} = a_{ij}' \in L^\infty(Q) \) \( \forall 1 \leq i, j \leq n \), we have \( |(A'(t) u, u)| \leq \sup_{Q} |a'_{ij}(x, t)| \| u \|_{H^1}^2 \); where we used Cauchy-Schwarz and Poincaré-Friedrichs inequalities. Then we obtain \( (A'(t) u, u) \leq \gamma \| u \|^2 \), where \( \| \cdot \| \) denote the norm in \( H_0^1(\Omega) \cap H^2(\Omega) \).

It is well known that \( H_0^1(\Omega) \cap H^2(\Omega) \hookrightarrow L^2(\Omega) \) compactly.

Let \( F : \mathbb{R} \to \mathbb{R} \) be the function defined by \( F(s) = s^3 \), and \( M : H_0^1(\Omega) \to L^2(\Omega) \) a operator defined by \( (Mu)(x) = F(u(x)) \).

Due to the properties of \( F \) it follows that \( M \) is monotone, hemicontinuous bounded and

\[ \int_0^t (M(u(s)), u'(s)) ds \geq -\sigma, \; \forall t \in [0, T] \]
for all $u \in E_c$ where $E_c$ is the set \{ $u \in L^\infty(0,T;H^1_0(\Omega))$, $u' \in L^2(0,T;L^2(\Omega))$ and \|$u(0)\|$ \leq C$ \}. The constant $\sigma$ depends on $C$.

Let us prove the two last properties. Being
\[
|M u|^2 = \int_\Omega |(M u)(x)|^2 dx = \int_\Omega |u^3(x)|^2 dx = \\
\int_\Omega |u(x)|^6 dx = \|u\|_{L^6(\Omega)}^6,
\]
it follows from Sobolev inequalities, $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$ with $\frac{1}{q} = \frac{1}{2} - \frac{1}{n}$ ($n \geq 3$). Therefore $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$ ($n = 3$) and, $|M u|^2 \leq c\|u\|^6$. So that, $M$ is bounded.

Let $g(\tau) = \int_0^\tau F(\tau) d\tau$. Then $g(\tau) \geq 0$, $\forall \tau \in \mathbb{R}$, and for $u \in E_c$,
\[
\int_0^t (M(u(s)),u'(s)) ds = \int_0^t \int_\Omega u^3(x,s) \frac{\partial u}{\partial s}(x,s) dx ds = \\
\int_0^t \int_\Omega F(u(x,s)) \frac{\partial u}{\partial s}(x,s) dx ds = \int_0^t \int_\Omega \frac{\partial g}{\partial s}(u(x,s)) dx ds = \\
= \int_\Omega g(u(x,t)) dx - \int_\Omega g(u(x,0)) dx \geq - \int_\Omega g(u(x,0)) dx = \\
= -\frac{1}{4} \int_\Omega [u(x,0)]^4 dx = -\frac{1}{4} \int_\Omega \|u(x,0)\|^4 dx \geq \\
\geq -\left[ \int_\Omega |u(x,0)|^2 dx \right]^{\frac{1}{2}} \cdot \left[ \int_\Omega \|u(x,0)\|^6 dx \right]^{\frac{1}{3}} \geq -\sigma.
\]
Therefore one has studied the existence and uniqueness of solutions of the mixed problem for the equation
\[
k_2(x)u'' + k_1(x)u' + A(t)u + u^3 = f.
\]

2) In the same scheme we have analogous results for the equations
\[
k_2(x)u'' + k_1(x)u' + A(t)u + M(u) = f
\]
where $(M u)(x) = F(u(x))$ here $F(s)$ is defined by
\[
F(s) = \begin{cases} 
\text{sign}(s) \frac{s^2}{1 + s^2} & \text{if } s \neq 0 \\
0 & \text{if } s = 0
\end{cases}
\]

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REFERENCES


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